HIGHER ALGEBRA

Classical

Revised Eighth Edition



S K Mapa

PREFACE TO THE FIRST EDITION

This book being a companion volume to the author's Higher Algebra (Abstract and Linear) is designed to serve as a text book for Mathematics Honours students. The subject matter developed in the book covers the syllabi of Honours Classical Algebra of different Universities.

The reader will find a modern approach to the treatment of the theory of Integers and Complex numbers which fall into the classical branch of Algebra. Some theorems on Inequalities have been dealt with elegant proofs. Each article is supplemented with illustrative examples which have been fully worked out for the benefit of the reader.

The author expresses his indebtedness to the celebrated authors of some standard texts on the subject which have been freely consulted.

The author also conveys his sincere thanks to the publisher and the printer for their co-operation in the publication of the book.

Inspite of best efforts some mistakes and misprints have crept in. An *errata* showing corrections for mistakes and major misprints is appended at the end.

Any correction and any suggestion for improvement of the book will be cordially accepted.

Calcutta
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S.K. MAPA

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1. INEQUALITIES

1.1. Introduction.

Throughout this chapter we are concerned with real numbers only.

When two real numbers are not equal, a relation of *inequality* is said to exist between them. The *property of trichotomy* in the set of all real numbers states that any two real numbers a, b must satisfy one and only one of the following relations–

(i) a is equal to b $\quad (a = b)$,
(ii) a is greater than b $\quad (a > b)$,
(iii) a is less than b $\quad (a < b)$.

The last two relations are inequality relations.

Therefore, if a be a real number different from 0, one of the following inequality relations must hold–

(i) $a > 0$, $\quad$ (ii) $a < 0$.

When $a > 0$, a is said to be *positive*; when $a < 0$, a is said to be *negative*.

We define $a > b$ if $a - b > 0$, and
$a < b$ if $a - b < 0$.

The relations $a > b$ and $b < a$ state the same inequality relation, since $a > 0 \Leftrightarrow -a < 0$.

The symbol $a \geq b$ means a is greater than or equal to b;
$a \leq b$ means a is less than or equal to b.

For example, if a is a real number then $a^2 \geq 0$;
if a is a positive real number then $a^2 > 0$.

1.2. Properties.

If a, b, c be real numbers, then

(i) $a > b$ and $b > c \Rightarrow a > c$,
(ii) $a > b \Rightarrow a + c > b + c$,
(iii) $a > b$ and $c > 0 \Rightarrow ac > bc$,
(iv) $a > b$ and $c < 0 \Rightarrow ac < bc$,
(v) $a > b$ and $c = 0 \Rightarrow ac = bc$.

Proof. (i) $a - c = (a - b) + (b - c)$
> 0, since $a - b > 0$ and $b - c > 0$.

Therefore $a > c$.

(ii) $(a + c) - (b + c) = a - b$
> 0, since $a > b$.

Therefore $a + c > b + c$.

(iii) $ac - bc = (a - b)c$
> 0, since $a - b > 0$ and $c > 0$.

Therefore $ac > bc$.

(iv) $ac - bc = (a - b)c$
< 0, since $a - b > 0$ and $c < 0$.

Therefore $ac < bc$.

(v) obvious.

Corollary. (i) $a \geq b$ and $b \geq c \Rightarrow a \geq c$,
(ii) $a \geq b$ and $b > c \Rightarrow a > c$,
(iii) $a \geq b \Rightarrow a + c \geq b + c$,
(iv) $a \geq b$ and $c > 0 \Rightarrow ac \geq bc$,
(v) $a \geq b$ and $c < 0 \Rightarrow ac \leq bc$.

Theorem 1.2.1. If $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be all real numbers such that $a_i > b_i$ for $i = 1, 2, \ldots, n$, then

$$a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n.$$

Proof. $(a_1 + a_2 + \cdots + a_n) - (b_1 + b_2 + \cdots + b_n)$
$= (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_n - b_n)$
> 0, since $a_i - b_i > 0$ for $i = 1, 2, \ldots, n$.
Therefore $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n$.

Theorem 1.2.2. If $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be all positive real numbers such that $a_i > b_i$ for $i = 1, 2, \ldots, n$, then

$$a_1 a_2 \ldots a_n > b_1 b_2 \ldots b_n.$$

Proof. $a_1 a_2 - b_1 b_2 = a_1(a_2 - b_2) + b_2(a_1 - b_1)$
> 0, since each term is positive.
Therefore $a_1 a_2 > b_1 b_2$.
Thus $a_1 > b_1$ and $a_2 > b_2 \Rightarrow a_1 a_2 > b_1 b_2$.
Similarly, $a_1 a_2 > b_1 b_2$ and $a_3 > b_3 \Rightarrow a_1 a_2 a_3 > b_1 b_2 b_3$.
Successive applications give $a_1 a_2 \ldots a_n > b_1 b_2 \ldots b_n$.

Note. The theorem does not hold if the numbers be not all positive.
For example, $5 > -2$ and $2 > -10$ but $5.2 < (-2).(-10)$.

Theorem 1.2.3. If a, b be positive real numbers with $a > b$, and n be a positive integer, then $a^n > b^n$.

Proof. The theorem holds for $n = 1$.

Let us assume that the theorem holds for a positive integer, say m. Then $a^m > b^m$.

Now $a^m > b^m > 0$ and $a > b > 0 \Rightarrow a^{m+1} > b^{m+1}$.

This shows that the theorem holds for $n = m + 1$ if it holds for $n = m$. And the theorem holds for $n = 1$.

By the principle of induction, the theorem holds for all positive integers n.

Note. If a, b are real numbers with $a > b$ and n is a positive integer, it does not necessarily follow that $a^n > b^n$.

For example, $2 > -3$ implies $(2)^2 < (-3)^2$, but $2 > -1$ implies $(2)^2 > (-1)^2$.

Theorem 1.2.4. If $a > b > 0$ and n be a negative integer, then $a^n < b^n$.

Proof. Let $n = -m$. Then m is a positive integer.

$\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab} < 0$, since $b - a < 0$ and $ab > 0$.

Therefore $\frac{1}{a} < \frac{1}{b}$.

Since m is a positive integer and $0 < \frac{1}{a} < \frac{1}{b}$, $(\frac{1}{a})^m < (\frac{1}{b})^m$, by the previous theorem.

Therefore $a^{-m} < b^{-m}$, i.e., $a^n < b^n$.

Theorem 1.2.5. If $a > b > 0$ and n be a rational number, then $a^n > b^n$ according as n is positive or negative.

Proof. When n is an integer, positive or negative, the theorem reduces to the theorems 1.2.3 and 1.2.4.

Case I. Let n be a positive fraction and $n = \frac{p}{q}$ where p, q are positive integers prime to each other and $q \neq 1$.

Let us consider two positive numbers $a^{1/q}$ and $b^{1/q}$ where $a^{1/q}$ denotes the positive qth root of the positive number a. (Such a positive root always exists, since a is positive.)

We assert that $a^{1/q} > b^{1/q}$. Because, $a^{1/q} \leq b^{1/q} \Rightarrow (a^{1/q})^q \leq (b^{1/q})^q \Rightarrow a \leq b$, a contradiction.

Since p is a positive integer, $a^{1/q} > b^{1/q} \Rightarrow (a^{1/q})^p > (b^{1/q})^p$, i.e., $a^n > b^n$.

Case II. Let n be negative fraction and $n = -m$ where m is positive.

$a > b > 0 \Rightarrow \frac{1}{b} > \frac{1}{a} > 0$

$\Rightarrow \left(\frac{1}{b}\right)^m > \left(\frac{1}{a}\right)^m$, by case 1

or, $b^{-m} > a^{-m}$, i.e., $a^n < b^n$.

This completes the proof.

Worked Examples.

1. If a, b, c be all real numbers prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

$a^2 + b^2 + c^2 - (ab + bc + ca) = \frac{1}{2}(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$ since each term is non-negative.

Therefore $a^2 + b^2 + c^2 \geq ab + bc + ca$, the equality occurs when $a = b = c$.

2. If a, b, c be all positive real numbers, prove that

$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \geq a + b + c$.

$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$.

Therefore $2(a^2 + b^2) \geq (a+b)^2$, the equality occurs when $a = b$

or, $\frac{a^2+b^2}{a+b} \geq \frac{a+b}{2}$, since $a + b > 0$.

Similarly, $\frac{b^2+c^2}{b+c} \geq \frac{b+c}{2}$, $\frac{c^2+a^2}{c+a} \geq \frac{c+a}{2}$.

Hence $\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \geq a + b + c$, by Theorem 1.2.1.

The equality occurs when $a = b = c$.

3. If a, b, c, d be all real numbers greater than 1, prove that

$(a+1)(b+1)(c+1)(d+1) < 8(abcd + 1)$.

$(a-1)(b-1) > 0$ since $a - 1 > 0, b - 1 > 0$

or, $ab + 1 > a + b$

or, $2(ab+1) > ab + 1 + a + b = (a+1)(b+1)$.

Therefore $(a+1)(b+1) < 2(ab+1)$ ... ... (i)

Similarly, $(c+1)(d+1) < 2(cd+1)$ ... ... (ii)

$ab > 1, cd > 1$. Using (i), we have

$(ab+1)(cd+1) < 2(abcd+1)$ ... ... (iii)

Hence $(a+1)(b+1)(c+1)(d+1) < 4(ab+1)(cd+1)$, by (i) and (ii); and $4(ab+1)(cd+1) < 8(abcd+1)$, by (iii)

Therefore $(a+1)(b+1)(c+1)(d+1) < 8(abcd+1)$.

4. If a, b, c be all positive real numbers, and n be a positive rational number, prove that

$a^n(a-b)(a-c) + b^n(b-a)(b-c) + c^n(c-a)(c-b) \geq 0$.

Case 1. Let $a = b = c$.

L.H.S. $= 0$, since each term is 0.

Case 2. Let two of a, b, c be equal. $a = b$, say.

L.H.S. $= c^n(c-a)^2 > 0$, since $c > 0, (c-a)^2 > 0$.

Case 3. No two of a, b, c are equal.

Without loss of generality, let $a > b > c$.

Then $a - b > 0, b - c > 0, a - c > 0$.

$a^n(a-b)(a-c) + b^n(b-a)(b-c) = (a-b)[a^n(a-c) - b^n(b-c)]$.

$a > b > c \Rightarrow a - c > b - c > 0$ and $a > b > 0 \Rightarrow a^n > b^n$, since $n > 0$. Therefore $a^n(a-c) > b^n(b-c)$, by Theorem 1.2.2.

Hence $a^n(a-b)(a-c) + b^n(b-a)(b-c) > 0$.

Also we have $c^n(c-a)(c-b) > 0$.

Therefore $a^n(a-b)(a-c) + b^n(b-a)(b-c) + c^n(c-a)(c-b) > 0$.

Combining all cases, we have

$a^n(a-b)(a-c) + b^n(b-a)(b-c) + c^n(c-a)(c-b) \geq 0$, the equality occurs when $a = b = c$.

1.3. Standard inequalities.

1. Weierstrass inequalities.

If $a_1, a_2, \ldots, a_n$ are all positive real numbers less than 1 and $s_n = a_1 + a_2 + \cdots + a_n$, then

$$1 - s_n < (1-a_1)(1-a_2)\ldots(1-a_n) < \frac{1}{1+s_n}$$

and $$1 + s_n < (1+a_1)(1+a_2)\ldots(1+a_n) < \frac{1}{1-s_n},$$

provided in the last inequality it is assumed that $s_n < 1$.

Proof. $(1-a_1)(1-a_2) = 1 - (a_1 + a_2) + a_1a_2$

$> 1 - (a_1 + a_2)$.

$(1-a_1)(1-a_2)(1-a_3) > [1 - (a_1 + a_2)](1 - a_3)$, since $1 - a_3 > 0$

$> 1 - (a_1 + a_2 + a_3)$.

Successive applications give

$(1-a_1)(1-a_2)\ldots(1-a_n) > 1 - (a_1 + a_2 + \cdots + a_n)$

i.e., $> 1 - s_n$.

In the same manner, $(1+a_1)(1+a_2)\ldots(1+a_n) > 1 + s_n$.

Since $0 < a_1 < 1$, $1 - a_1^2 < 1$.
Therefore $1 - a_1 < \frac{1}{1+a_1}$, since $1 + a_1 > 0$.
Similarly, $1 - a_2 < \frac{1}{1+a_2}, \ldots, 1 - a_n < \frac{1}{1+a_n}$.
Therefore $(1 - a_1)(1 - a_2)\ldots(1 - a_n) < \frac{1}{(1+a_1)(1+a_2)\ldots(1+a_n)}$
$< \frac{1}{1+s_n}$.

Again, $1 - a_1^2 < 1$.
Therefore $1 + a_1 < \frac{1}{1-a_1}$, since $1 - a_1 > 0$.
Similarly, $1 + a_2 < \frac{1}{1-a_2}, \cdots, 1 + a_n < \frac{1}{1-a_n}$.
Therefore $(1 + a_1)(1 + a_2)\ldots(1 + a_n) < \frac{1}{(1-a_1)(1-a_2)\ldots(1-a_n)}$
$< \frac{1}{1-s_n}$, since $1 - s_n > 0$.

This completes the proof.

Worked Example.

1. If n be a positive integer, prove that $\frac{1.3.7\ldots(2^n-1)}{2.4.8\ldots2^n} < \frac{2^n}{2^{n+1}-1}$.

By Weierstrass inequality,
$(1 - \frac{1}{2})(1 - \frac{1}{2^2})\ldots(1 - \frac{1}{2^n}) < \frac{1}{1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}$
$= \frac{2^n}{2^{n+1}-1}$
or, $\frac{1.3.7\ldots(2^n-1)}{2.4.8\ldots2^n} < \frac{2^n}{2^{n+1}-1}$

The ordered array of n real numbers $a_1, a_2, \ldots, a_n$ is denoted by $(a_1, a_2, \ldots, a_n)$. Let $(a) = (a_1, a_2, \ldots, a_n), (b) = (b_1, b_2, \ldots, b_n)$. Then $(a) = (b)$ if $a_i = b_i$ for $i = 1, 2, \ldots, n$. The ordered arrays (a) and (b) are said to be *proportional* if there exists a non-zero real number k such that $a_i = kb_i$ for $i = 1, 2, \ldots, n$.

For example, $(1, 1, 2, 0, -1)$ and $(2, 2, 4, 0, -2)$ are proportional.

2. Cauchy-Schwarz inequality.

If $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be all real numbers, then $(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \geq (a_1b_1+a_2b_2+\cdots+a_nb_n)^2$, the equality occurs when either

(i) $a_i = 0$ for $i = 1, 2, \ldots, n$; or $b_i = 0$ for $i = 1, 2, \ldots, n$; or both $a_i = 0$ and $b_i = 0$ for $i = 1, 2, \ldots, n$;

or (ii) $a_i = kb_i$ for some non-zero real $k, i = 1, 2, \ldots, n$.

Proof. **Case I.** If $a_i = 0$ for $i = 1, 2, \ldots, n$; or $b_i = 0$ for $i = 1, 2, \ldots, n$; or both $a_i = 0$ and $b_i = 0$ for $i = 1, 2, \ldots, n$; then the equality holds, each side being zero.

Case II. Let not all of a_i and not all of b_i be zero.

Sub-case (i). Let $a_i = kb_i$ for some non zero real $k, i = 1, 2, \ldots, n$.

Then $(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) = k^2(b_1^2+b_2^2+\cdots+b_n^2)^2$ and $(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 = k^2(b_1^2 + b_2^2 + \cdots + b_n^2)^2$.

Therefore the equality holds in this sub-case.

Sub-case (ii). Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be not proportional. Let us consider the expression

$(a_1 - \lambda b_1)^2 + (a_2 - \lambda b_2)^2 + \cdots + (a_n - \lambda b_n)^2$, where λ is real.

For all real λ, the expression ≥ 0. The equality occurs only when
$a_1 - \lambda b_1 = 0, a_2 - \lambda b_2 = 0, \cdots, a_n - \lambda b_n = 0$
i.e., when $(a_1, a_2, \ldots, a_n) = \lambda(b_1, b_2, \ldots, b_n)$
i.e., when $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are proportional.

Therefore, in this sub-case $(a_1 - \lambda b_1)^2 + (a_2 - \lambda b_2)^2 + \cdots + (a_n - \lambda b_n)^2 > 0$ for all real λ.

or, $(a_1^2 + a_2^2 + \cdots + a_n^2) - 2\lambda(a_1b_1 + a_2b_2 + \cdots + a_nb_n) + \lambda^2(b_1^2 + b_2^2 + \cdots + b_n^2) > 0$

or, $B\lambda^2 - 2C\lambda + A > 0$, where $A = a_1^2 + a_2^2 + \cdots + a_n^2$, $B = b_1^2 + b_2^2 + \cdots + b_n^2$, $C = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

The roots of the equation $Bx^2 - 2Cx + A = 0$ must be imaginary, because otherwise, there would exist some real λ for which the equality $B\lambda^2 - 2C\lambda + A = 0$ would hold, a contradiction.

Therefore $AB > C^2$
or, $(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) > (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$.
This completes the proof.

Note. In particular, if $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be all positive real numbers, then
$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \geq (a_1b_1+a_2b_2+\cdots+a_nb_n)^2$, the equality occurs if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Worked Examples (continued).

2. For all real x, y, prove that $-\frac{1}{2} \leq \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \leq \frac{1}{2}$.

Let us consider ordered pairs $(2x, 1 - x^2)$ and $(1 - y^2, 2y)$.

By Cauchy-Schwarz inequality,
$[2x(1-y^2)+(1-x^2)2y]^2 \le [(2x)^2+(1-x^2)^2][(1-y^2)^2+(2y)^2]$
or, $[2(x+y)(1-xy)]^2 \le (1+x^2)^2(1+y^2)^2$.
or, $[\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}]^2 \le \frac{1}{4}$
or, $-\frac{1}{2} \le \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \le \frac{1}{2}$.

3. If $a_i > -\frac{1}{3}$, $(i = 1, 2, 3)$ and $a+b+c = 1$, prove that $\sqrt{3a+1}+\sqrt{3b+1}+\sqrt{3c+1} \le 3\sqrt{2}$.

Let us consider ordered triplets $(1,1,1)$ and $(\sqrt{3a+1}, \sqrt{3b+1}, \sqrt{3c+1})$.
By Cauchy-Schwarz inequality,
$(\sqrt{3a+1}+\sqrt{3b+1}+\sqrt{3c+1})^2 \le (1+1+1)[(3a+1)+(3b+1)+(3c+1)]$.
or, $\sqrt{3a+1}+\sqrt{3b+1}+\sqrt{3c+1} \le 3\sqrt{2}$.
The equality occurs if $3a+1 = 3b+1 = 3c+1$, i.e., if $a = b = c$.

4. If $a, b, c, d > 0$ and $a+b+c+d = 1$, prove that $\frac{a}{1+b+c+d} + \frac{b}{1+a+c+d} + \frac{c}{1+a+b+d} + \frac{d}{1+a+b+c} \ge \frac{4}{7}$.

$\frac{a}{1+b+c+d} + \frac{b}{1+a+c+d} + \frac{c}{1+a+b+d} + \frac{d}{1+a+b+c}$
$= \frac{a}{2-a} + \frac{b}{2-b} + \frac{c}{2-c} + \frac{d}{2-d} = \frac{2}{2-a} + \frac{2}{2-b} + \frac{2}{2-c} + \frac{2}{2-d} - 4$.
$= 2[\frac{1}{2-a} + \frac{1}{2-b} + \frac{1}{2-c} + \frac{1}{2-d}] - 4$

Let us consider positive numbers $\frac{1}{\sqrt{2-a}}, \frac{1}{\sqrt{2-b}}, \frac{1}{\sqrt{2-c}}, \frac{1}{\sqrt{2-d}}$ and $\sqrt{2-a}, \sqrt{2-b}, \sqrt{2-c}, \sqrt{2-d}$.

By Cauchy-Schwarz inequality,
$(1+1+1+1)^2 \le [\frac{1}{2-a} + \frac{1}{2-b} + \frac{1}{2-c} + \frac{1}{2-d}][8-(a+b+c+d)]$
or, $\frac{1}{2-a} + \frac{1}{2-b} + \frac{1}{2-c} + \frac{1}{2-d} \ge \frac{16}{7}$.

Therefore $\frac{a}{1+b+c+d} + \frac{b}{1+a+c+d} + \frac{c}{1+a+b+d} + \frac{d}{1+a+b+c} \ge \frac{32}{7} - 4$, i.e., $\ge \frac{4}{7}$.

The equality occurs when $2-a = 2-b = 2-c = 2-d$, i.e., when $a = b = c = d$.

5. If $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$; $c_1, c_2, \ldots, c_n$ be all positive real numbers prove that

$(a_1b_1c_1 + a_2b_2c_2 + \cdots + a_nb_nc_n)^2 < (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)(c_1^2 + c_2^2 + \cdots + c_n^2)$.

Let $d_i = b_ic_i$, $i = 1, 2, \ldots, n$.
By Cauchy-Schwarz inequality, $(a_1d_1 + a_2d_2 + \cdots + a_nd_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(d_1^2 + d_2^2 + \cdots + d_n^2)$.
Again, $(d_1^2 + d_2^2 + \cdots + d_n^2) = b_1^2c_1^2 + b_2^2c_2^2 + \cdots + b_n^2c_n^2 < (b_1^2 + b_2^2 + \cdots + b_n^2)(c_1^2 + c_2^2 + \cdots + c_n^2)$,
since b_i, c_i are all positive.
Therefore $(a_1b_1c_1 + a_2b_2c_2 + \cdots + a_nb_nc_n)^2 < (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)(c_1^2 + c_2^2 + \cdots + c_n^2)$.

Theorem 1.3.1. If $a_1, a_2, \ldots, a_n$ be n positive real numbers, not all equal, and p, q are rational numbers, then
$\frac{a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}}{n} >$ or $< \frac{a_1^p + a_2^p + \cdots + a_n^p}{n} \cdot \frac{a_1^q + a_2^q + \cdots + a_n^q}{n}$
according as p and q have the same sign or opposite signs.

Proof. **Case 1.** Let p, q have the same sign. Let i and j be any two of the numbers $1, 2, \ldots, n$. Since p, q are of the same sign, $a_i^p - a_j^p$ and $a_i^q - a_j^q$ are both positive or both negative or both zero.
Therefore $(a_i^p - a_j^p)(a_i^q - a_j^q) \ge 0$
or, $a_i^{p+q} + a_j^{p+q} \ge a_i^p a_j^q + a_i^q a_j^p$.
There are ${}^nC_2$ relations of this type, not of all them are equalities.
Adding, we have
$(n-1)(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) > \Sigma a_i^p a_j^q$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$; $i \ne j$
or, $n(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) > (a_1^p + a_2^p + \cdots + a_n^p)(a_1^q + a_2^q + \cdots + a_n^q)$
or, $\frac{a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}}{n} > \frac{a_1^p + a_2^p + \cdots + a_n^p}{n} \cdot \frac{a_1^q + a_2^q + \cdots + a_n^q}{n}$.

Case 2. Let p, q have opposite signs. Then $a_i^p - a_j^p$ and $a_i^q - a_j^q$ have opposite signs when $a_i \ne a_j$ and both are zero when $a_i = a_j$.
Therefore $(a_i^p - a_j^p)(a_i^q - a_j^q) \le 0$.
Proceeding with similar arguments as in case 1, we can prove
$\frac{a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}}{n} < \frac{a_1^p + a_2^p + \cdots + a_n^p}{n} \cdot \frac{a_1^q + a_2^q + \cdots + a_n^q}{n}$.
This completes the proof.

Note. The theorem can be generalised. If $a_1, a_2, \ldots, a_n$ be n positive real numbers, not all equal, and $p_1, p_2, \ldots, p_m$ be m rational numbers, all positive or all negative, such that $s = p_1 + p_2 + \cdots + p_m$, then

$$\frac{a_1^s+a_2^s+\cdots+a_n^s}{n} > \frac{a_1^{p_1}+a_2^{p_1}+\cdots+a_n^{p_1}}{n}\cdot\frac{a_1^{p_2}+a_2^{p_2}+\cdots+a_n^{p_2}}{n}\cdots\frac{a_1^{p_m}+a_2^{p_m}+\cdots+a_n^{p_m}}{n}.$$

Taking in particular, $p_1 = p_2 = \cdots = p_m = 1$, we have

$$\frac{a_1^m+a_2^m+\cdots+a_n^m}{n} > \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m.$$

Worked Example (continued).

6. If a, b, c, d be positive real numbers, not all equal, prove that
$(a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}) > 16$.

We have $\frac{a^p+b^p+c^p+d^p}{4}\cdot\frac{a^q+b^q+c^q+d^q}{4} > \frac{a^{p+q}+b^{p+q}+c^{p+q}+d^{p+q}}{4}$, where p, q are rational numbers of opposite signs.

Let $p = 1, q = -1$. Then $\frac{a+b+c+d}{4}\cdot\frac{(a^{-1}+b^{-1}+c^{-1}+d^{-1})}{4} > 1$

or, $(a+b+c+d)(a^{-1}+b^{-1}+c^{-1}+d^{-1}) > 16$.

Another method.

Let us consider real numbers $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$ and $\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}$.

By Cauchy-Schwarz inequality,
$$(1+1+1+1)^2 \le (a+b+c+d)(\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}+\tfrac{1}{d}).$$
The equality occurs when $a = b = c = d$.

Since here a, b, c, d are not all equal, $(a+b+c+d)(a^{-1}+b^{-1}+c^{-1}+d^{-1}) > 16$.

Exercises 1A

1. If $a_1, a_2, \ldots, a_n$ be n positive real numbers in ascending order of magnitude, prove that $a_1 < \frac{a_1^2+a_2^2+\cdots+a_n^2}{a_1+a_2+\cdots+a_n} < a_n$.

2. If $a_1, a_2, \ldots, a_n$ be real numbers, not all equal, and $p_1, p_2, \ldots, p_n$ be positive real numbers, prove that

$$\min(a) < \frac{p_1a_1+p_2a_2+\cdots+p_na_n}{p_1+p_2+\cdots+p_n} < \max(a),$$

where $\min(a)$ and $\max(a)$ denote respectively the least and the greatest of the numbers $a_1, a_2, \ldots, a_n$.

3. If $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be all real numbers and $b_i > 0$ for $i = 1, 2, \ldots, n$ prove that $m \le \frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n} \le M$,

where $m = \min\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots, \frac{a_n}{b_n}\}$, $M = \max\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots, \frac{a_n}{b_n}\}$.

4. If $x_1, x_2, \ldots, x_n$ be n real numbers satisfying $0 < x_1 < x_2 \cdots < x_n < \frac{\pi}{2}$, prove that

$$\tan x_1 < \frac{\sin x_1+\sin x_2+\cdots+\sin x_n}{\cos x_1+\cos x_2+\cdots+\cos x_n} < \tan x_n.$$

5. If a, b, c be real numbers, prove that
$(a+b-c)^2+(b+c-a)^2+(c+a-b)^2 \ge ab+bc+ca$.

6. If a, b, c be positive real numbers such that the sum of any two is greater than the third, prove that

(i) $a^2(p-q)(p-r)+b^2(q-p)(q-r)+c^2(r-p)(r-q) \ge 0$ for all real p, q, r;

(ii) $a^2yz+b^2zx+c^2xy \le 0$ for all real x, y, z such that $x+y+z = 0$.

7. If a, b, c, d be positive real numbers, each less than 1, prove that
$8(abcd+1) > (a+1)(b+1)(c+1)(d+1)$.

8. If a, b, c be positive real numbers, not all equal, and n is a negative rational number, prove that
$a^n(a-b)(a-c)+b^n(b-a)(b-c)+c^n(c-a)(c-b) > 0$.

9. If n be a positive integer greater than 2, prove that $(n!)^2 > n^n$.
[**Hint.** $(r-1)(n-r) > 0$ if $1 < r < n$. Therefore $r(n-r+1) > n$.]

10. If $a_1, a_2, \ldots, a_n$ be n positive real numbers ($n > 2$) in arithmetic progression, prove that $a_1a_2\ldots a_n > (a_1a_n)^{n/2}$.

11. If a, b, x, y be positive real numbers, prove that $\frac{ax+by}{a+b} \ge \frac{(a+b)xy}{ay+bx}$.

12. Prove that $1!.3!.5!\ldots(2n-1)! > (n!)^n$.

13. If a, b, c be positive real numbers, not all equal, prove that

(i) $2(a^3+b^3+c^3) > a^2(b+c)+b^2(c+a)+c^2(a+b) > 6abc$;

(ii) $\frac{b+c}{b^2+c^2}+\frac{c+a}{c^2+a^2}+\frac{a+b}{a^2+b^2} < \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$;

(iii) $(a^3+b^3+c^3)(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}) > 9$;

(iv) $\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} < \frac{a+b+c}{2}$.

14. If a, b, c, x, y, z be all real numbers and $a^2+b^2+c^2 = 1, x^2+y^2+z^2 = 1$, prove that $-1 \le ax+by+cz \le 1$.

15. If a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; d_1, d_2, d_3 be all real numbers, prove that $(a_1b_1c_1d_1+a_2b_2c_2d_2+a_3b_3c_3d_3)^4 \le (a_1^4+a_2^4+a_3^4)(b_1^4+b_2^4+b_3^4)(c_1^4+c_2^4+c_3^4)(d_1^4+d_2^4+d_3^4)$.

1.4. Arithmetic, Geometric and Harmonic means.

Let $a_1, a_2, \ldots, a_n$ be n *positive* real numbers.

The *arithmetic mean* (*A.M.*) of the numbers is defined by $\frac{a_1+a_2+\cdots+a_n}{n}$ and is denoted by A.

The *geometric mean* (*G.M.*) of the numbers is defined by $\sqrt[n]{(a_1a_2\ldots a_n)}$ and is denoted by G.

The *harmonic mean* (*H.M.*) of the numbers is defined by $\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$ and is denoted by H.

Let $p_1, p_2, \ldots, p_n$ be n positive rational numbers.

The *weighted arithmetic mean* of $a_1, a_2, \ldots, a_n$ with associated weights $p_1, p_2, \ldots, p_n$ respectively is defined by

$$\frac{p_1a_1++p_2a_2+\cdots+p_na_n}{p_1+p_2+\cdots+p_n}$$

and is denoted by $A(a,p)$.

The *weighted geometric mean* of $a_1, a_2, \ldots, a_n$ with associaed weights $p_1, p_2, \ldots, p_n$ respectively is defined by

$$(a_1^{p_1}a_2^{p_2}\ldots a_n^{p_n})^{1/(p_1+p_2+\cdots+p_n)}$$

and is denoted by $G(a,p)$.

The *weighted harmonic mean* of $a_1, a_2, \ldots, a_n$ with associated weights $p_1, p_2, \ldots, p_n$ respectively is defined by

$$\frac{p_1+p_2+\cdots+p_n}{\frac{p_1}{a_1}+\frac{p_2}{a_2}+\cdots+\frac{p_n}{a_n}}$$

and is denoted by $H(a,p)$.

Theorem 1.4.1. If $a_1, a_2, \ldots, a_n$ be n be positive real numbers then $\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{(a_1a_2\ldots a_n)}$.

The equality occurs when $a_1 = a_2 = \cdots = a_n$.

The thoerem states that the arithmetic mean of n positive real numbers is greater than or equal to their geometric mean, the equality occurs when the numbers are all equal.

Proof. $a_1a_2 = \left(\frac{a_1+a_2}{2}\right)^2 - \left(\frac{a_1-a_2}{2}\right)^2$

$\leq \left(\frac{a_1+a_2}{2}\right)^2$, since $\left(\frac{a_1-a_2}{2}\right)^2 \geq 0$. ... ... (i)

The equality occurs when $\left(\frac{a_1-a_2}{2}\right)^2 = 0$, i.e., when $a_1 = a_2$.

Similarly, $a_3a_4 \leq \left(\frac{a_3+a_4}{2}\right)^2$, the equality occurs when $a_3 = a_4$.

Therefore $a_1a_2a_3a_4 \leq \left(\frac{a_1+a_2}{2}\right)^2\left(\frac{a_3+a_4}{2}\right)^2$. ... ... (ii)

But $\frac{a_1+a_2}{2}, \frac{a_3+a_4}{2} \leq \left(\frac{a_1+a_2+a_3+a_4}{4}\right)^2$, from (i). The equality occurs when $\frac{a_1+a_2}{2} = \frac{a_3+a_4}{2}$.

Therefore $a_1a_2a_3a_4 \leq \left(\frac{a_1+a_2+a_3+a_4}{4}\right)^4$, from (ii). The equality occurs when $a_1 = a_2, a_3 = a_4$, $\frac{a_1+a_2}{2} = \frac{a_3+a_4}{2}$, i.e., when $a_1 = \cdots = a_4$.

Similarly, $a_5a_6a_7a_8 \leq \left(\frac{a_5+a_6+a_7+a_8}{4}\right)^4$.

Proceeding with similar arguments we have

$a_1a_2\ldots a_8 \leq \left(\frac{a_1+a_2+\cdots+a_8}{8}\right)^8$, the equality occurs when $a_1 = a_2 = \cdots = a_8$.

Continuing thus, when $n = 2^m$, where m is a positive integer,

$a_1a_2\ldots a_n \leq \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n$, the equality occurs when $a_1 = a_2 = \cdots = a_n$.

Let us consider the case when n is not a power of 2. Then $2^{m-1} < n < 2^m$ for some positive integer $m > 1$. In this case there exists a positive integer p such that $n + p = 2^m$.

Let us consider $n+p$ positive numbers $a_1, a_2, \ldots, a_n, a, a, \ldots, a$ where a is repeated p times and $a = \frac{a_1+a_2+\cdots+a_n}{n}$.

Since $n + p = 2^m$, by what we proved,

$a_1a_2\ldots a_n.a^p \leq \left(\frac{a_1+a_2+\cdots+a_n+pa}{n+p}\right)^{n+p}$, the equality occurs when $a_1 = a_2 = \cdots = a_n = a$, i.e., when $a_1 = a_2 = \cdots = a_n$.

or, $a_1a_2\ldots a_n.a^p \leq \left(\frac{na+pa}{n+p}\right)^{n+p}$

or, $a_1a_2\ldots a_n.a^p \leq a^{n+p}$

or, $a_1a_2\ldots a_n \leq a^n$

or, $a_1a_2\ldots a_n \leq \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n$

or, $\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{(a_1a_2\ldots a_n)}$, the equality occurs when $a_1 = a_2 = \cdots = a_n$.

This completes the proof.

Corollary. The arithmetic mean of n positive real numbers, not all equal, is greater than their geometric mean.

Theorem 1.4.2. If $a_1, a_2, \ldots, a_n$ be n be positive real numbers and G, H be their geometric mean and harmonic mean respectively, then $G \geq H$. The equality occurs when $a_1 = a_2 = \cdots = a_n$.

Proof. Since $a_1, a_2, \ldots, a_n$ are positive, $\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}$ are all positive.

The arithmetic mean of $\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}$ is $\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}$ and the geometric mean of $\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}$ is $(\frac{1}{a_1}\cdot\frac{1}{a_2}\cdots\frac{1}{a_n})^{1/n}$.

By the previous theorem, we have

$\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n} \geq (\frac{1}{a_1}\cdot\frac{1}{a_2}\cdots\frac{1}{a_n})^{1/n}$, the equality occurs when $\frac{1}{a_1} = \frac{1}{a_2} = \cdots = \frac{1}{a_n}$, i.e., when $a_1 = a_2 = \cdots = a_n$.

or, $(a_1 a_2 \ldots a_n)^{1/n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$.

That is, $G \geq H$, the equality occurs when $a_1 = a_2 = \cdots = a_n$.

This completes the proof.

Corollary. The geometric mean of n positive real numbers, not all equal, is greater than their harmonic mean.

Theorem 1.4.3. If $a_1, a_2, \ldots, a_n$ be n positive real numbers and $p_1, p_2, \ldots, p_n$ be n positive rational numbers, then

$$\frac{p_1a_1+p_2a_2+\cdots+p_na_n}{p_1+p_2+\cdots+p_n} \geq (a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n})^{1/(p_1+p_2+\cdots+p_n)},$$

the equality occurs when $a_1 = a_2 = \cdots = a_n$.

[The weighted arithmetic mean of n positive real numbers is greater than or equal to their weighted geometric mean, the equality occurs when the numbers are all equal.]

Proof. **Case I.** Let $p_1, p_2, \ldots, p_n$ be all positive integers. Let us consider $p_1+p_2+\cdots+p_n$ positive numbers in which a_1 occurs p_1 times, a_2 occurs p_2 times, ..., a_n occurs p_n times.

The arithmetic mean of these is $\frac{p_1a_1+p_2a_2+\cdots+p_na_n}{p_1+p_2+\cdots+p_n}$ and the geometric mean of these is $(a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n})^{1/(p_1+p_2+\cdots+p_n)}$.

Therefore $\frac{p_1a_1+p_2a_2+\cdots+p_na_n}{p_1+p_2+\cdots+p_n} \geq (a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n})^{1/(p_1+p_2+\cdots+p_n)}$,

the equality occurs when $a_1 = a_2 = \cdots = a_n$.

Case II. Let $p_1, p_2, \ldots, p_n$ be positive rational numbers, not all of which are integers.

Let ρ be the l.c.m. of the denominators of $p_1, p_2, \ldots, p_n$. Then $\rho p_1, \rho p_2, \ldots, \rho p_n$ are all positive integers.

By Case 1,

$$\frac{\rho p_1a_1+\rho p_2a_2+\cdots+\rho p_na_n}{\rho p_1+\rho p_2+\cdots+\rho p_n} \geq (a_1^{\rho p_1}a_2^{\rho p_2}\cdots a_n^{\rho p_n})^{1/(\rho p_1+\rho p_2+\cdots+\rho p_n)},$$

the equality occurs when $a_1 = a_2 = \cdots = a_n$

or, $\frac{p_1a_1+p_2a_2+\cdots+p_na_n}{p_1+p_2+\cdots+p_n} \geq (a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n})^{1/(p_1+p_2+\cdots+p_n)}$, the equality occurs when $a_1 = a_2 = \cdots = a_n$.

This completes the proof.

Corollary. If $a_1, a_2, \ldots, a_n$ be n positive real numbers and $q_1, q_1, \ldots, q_n$ be n positive rational numbers such that $q_1 + q_2 + \cdots + q_n = 1$, then $q_1a_1 + q_2a_2 + \cdots + q_na_n \geq a_1^{q_1}a_2^{q_2}\ldots a_n^{q_n}$.

Theorem 1.4.4. If $a_1, a_2, \ldots, a_n$ be n positive real numbers and $p_1, p_2, \ldots, p_n$ be n positive rational numbers, then

$$(a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n})^{1/(p_1+p_2+\cdots+p_n)} \geq \frac{p_1+p_2+\cdots+p_n}{\frac{p_1}{a_1}+\frac{p_2}{a_2}+\cdots+\frac{p_n}{a_n}},$$

the equality occurs when $a_1 = a_2 = \cdots = a_n$.

Proof. Since $a_1, a_2, \ldots, a_n$ are positive, $\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}$ are all positive.

By the previous theorem,

$\frac{p_1\cdot\frac{1}{a_1}+p_2\cdot\frac{1}{a_2}+\cdots+p_n\cdot\frac{1}{a_n}}{p_1+p_2+\cdots+p_n} \geq [(\frac{1}{a_1})^{p_1}(\frac{1}{a_2})^{p_2}\cdots(\frac{1}{a_n})^{p_n}]^{1/(p_1+p_2+\cdots+p_n)}$, the equality occurs when $\frac{1}{a_1} = \frac{1}{a_2} = \cdots = \frac{1}{a_n}$, i.e.,when $a_1 = a_2 = \cdots = a_n$

or, $\frac{\frac{p_1}{a_1}+\frac{p_2}{a_2}+\cdots+\frac{p_n}{a_n}}{p_1+p_2+\cdots+p_n} \geq [\frac{1}{a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}}]^{1/(p_1+p_2+\cdots+p_n)}$

or, $(a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n})^{1/(p_1+p_2+\cdots+p_n)} \geq \frac{p_1+p_2+\cdots+p_n}{\frac{p_1}{a_1}+\frac{p_2}{a_2}+\cdots+\frac{p_n}{a_n}}$, the equality occurs when $a_1 = a_2 = \cdots = a_n$.

This completes the proof.

Worked Examples.

1. If a, b, c, d are positive real numbers, not all equal, prove that

$$a^5 + b^5 + c^5 + d^5 > abcd(a + b + c + d).$$

We have $\frac{a^{p+q}+b^{p+q}+c^{p+q}+d^{p+q}}{4} > \frac{a^p+b^p+c^p+d^p}{4}\cdot\frac{a^q+b^q+c^q+d^q}{4}$ if p, q are rational numbers of the same sign.

Taking $p = 4, q = 1$, we have $\frac{a^5+b^5+c^5+d^5}{4} > \frac{a^4+b^4+c^4+d^4}{4}\cdot\frac{a+b+c+d}{4}$.

But $\frac{a^4+b^4+c^4+d^4}{4} > \sqrt[4]{(a^4b^4c^4d^4)}$ [A.M. > G.M.

$= abcd.$

Therefore $a^5 + b^5 + c^5 + d^5 > abcd(a + b + c + d)$.

2. If n be a positive integer > 1, prove that $n(n+1)^2 > 4(n!)^{3/n}$.

Let us consider n unequal numbers $1^3, 2^3, \ldots, n^3$.

Applying A.M.>G.M., we have $\frac{1^3+2^3+\cdots+n^3}{n} > (1^3.2^3\ldots n^3)^{1/n}$

or, $\frac{n^2(n+1)^2}{4n} > (n!)^{3/n}$

or, $n(n+1)^2 > 4(n!)^{3/n}$.

Note. If $n = 1$ the equality reduces to an equality.

3. If x, y, z are positive real numbers and $x + y + z = 1$, prove that

$$8xyz \le (1-x)(1-y)(1-z) \le \tfrac{8}{27}.$$

$1 - x = y + z, 1 - y = z + x, 1 - z = x + y.$

We have $\frac{y+z}{2} \ge \sqrt{yz}$, $\frac{z+x}{2} \ge \sqrt{zx}$, $\frac{x+y}{2} \ge \sqrt{xy}$.

Therefore $(y+z)(z+x)(x+y) \ge 8xyz$, the equality occurs when $x = y = z$

or, $(1-x)(1-y)(1-z) \ge 8xyz$.

Again let us consider three positive real numbers $1-x, 1-y, 1-z$. Applying A.M.$\ge$ G.M., we have

$\frac{(1-x)+(1-y)+(1-z)}{3} \ge \sqrt[3]{(1-x)(1-y)(1-z)}$, the equality occurs when $1-x = 1-y = 1-z$, i.e., when $x = y = z$.

or, $(\frac{3-1}{3})^3 \ge (1-x)(1-y)(1-z)$

or, $\frac{8}{27} \ge (1-x)(1-y)(1-z)$.

Therefore $8xyz \le (1-x)(1-y)(1-z) \le \frac{8}{27}$, the equality occurs when $x = y = z = \frac{1}{3}$.

4. If a, b, c be positive real numbers such that the sum of any two is greater than the third, prove that

$$abc \ge (a+b-c)(b+c-a)(c+a-b).$$

Let us consider two positive numbers $a+b-c, b+c-a$.

$$\frac{(a+b-c)+(b+c-a)}{2} \ge \sqrt{(a+b-c)(b+c-a)}$$

$$\text{or, } b \ge \sqrt{(a+b-c)(b+c-a)},$$

Similarly, $c \ge \sqrt{(b+c-a)(c+a-b)}$,

$$a \ge \sqrt{(c+a-b)(a+b-c)}.$$

Therefore $abc \ge (a+b-c)(b+c-a)(c+a-b)$, the equality occurs when $b+c-a = c+a-b = a+b-c$, i.e., when $a = b = c$.

5. If n be a positive integer, prove that

$$\frac{1}{\sqrt{4n+1}} < \frac{3.7.11....(4n-1)}{5.9.13....(4n+1)} < \sqrt{\frac{3}{4n+3}}.$$

If r be a positive integer, we have

$$\frac{(4r+1)+(4r-3)}{2} > \sqrt{(4r+1)(4r-3)}$$

or, $4r - 1 > \sqrt{(4r+1)(4r-3)}$

or, $\frac{4r-1}{4r+1} > \sqrt{\frac{4r-3}{4r+1}}$.

Taking $r = 1, 2, 3, \ldots, n$ and multiplying, we have

$$\frac{3.7.11....(4n-1)}{5.9.13....(4n+1)} > \frac{1}{\sqrt{4n+1}}.$$

Again, if r be a positive integer, we have

$$\frac{(4r-1)+(4r+3)}{2} > \sqrt{(4r-1)(4r+3)}$$

or, $\frac{4r+1}{4r-1} > \sqrt{\frac{4r+3}{4r-1}}$.

Taking $r = 1, 2, 3, \ldots, n$ and multiplying, we have

$$\frac{5.9.13....(4n+1)}{3.7.11....(4n-1)} > \sqrt{\frac{4n+3}{3}}$$

or, $\frac{3.7.11....(4n-1)}{5.9.13....(4n+1)} < \sqrt{\frac{3}{4n+3}}$.

Therefore $\frac{1}{\sqrt{4n+1}} < \frac{3.7.11....(4n-1)}{5.9.13....(4n+1)} < \sqrt{\frac{3}{4n+3}}$.

6. If a, b, c be positive real numbers and $abc = k^3$, prove that

$$(1+a)(1+b)(1+c) \ge (1+k)^3.$$

$(1+a)(1+b)(1+c) = 1 + \Sigma a + \Sigma ab + abc.$

We have $\frac{\Sigma a}{3} \ge \sqrt[3]{abc}$ and $\frac{\Sigma ab}{3} \ge \sqrt[3]{a^2b^2c^2}$, the equality occurs in both cases when $a = b = c$. That is, $\Sigma a \ge 3k$ and $\Sigma ab \ge 3k^2$.

Therefore $(1+a)(1+b)(1+c) \ge 1 + 3k + 3k^2 + k^3$

or, $(1+a)(1+b)(1+c) \ge (1+k)^3$, the equality occurs when $a = b = c$.

7. If $a_1, a_2, \ldots, a_5$ be positive real numbers, prove that

$$\left(\frac{a_1+a_2+\cdots+a_5}{5}\right)^5 \ge \left(\frac{a_1+a_2}{2}\right)^2 \left(\frac{a_3+a_4+a_5}{3}\right)^3$$

Let us consider two positive numbers $\frac{a_1+a_2}{2}$, $\frac{a_3+a_4+a_5}{3}$ with associated weights 2 and 3 respectively.

Then $\frac{2.\frac{a_1+a_2}{2} + 3.\frac{a_3+a_4+a_5}{3}}{2+3} \ge \left[\left(\frac{a_1+a_2}{2}\right)^2 \left(\frac{a_3+a_4+a_5}{3}\right)^3\right]^{1/5}$, the equality occurs when $\frac{a_1+a_2}{2} = \frac{a_3+a_4+a_5}{3}$

or, $\left(\frac{a_1+a_2+\cdots+a_5}{5}\right)^5 \ge \left(\frac{a_1+a_2}{2}\right)^2 \left(\frac{a_3+a_4+a_5}{3}\right)^3$, the equality occurs when $\frac{a_1+a_2}{2} = \frac{a_3+a_4+a_5}{3}$.

8. If $a_1, a_2, \ldots, a_n$ be n positive real numbers, prove that

$$\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n \geq a_n \left(\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}\right)^{n-1}.$$

Let $b = \frac{a_1+a_2+\cdots+a_{n-1}}{n-1}$. Let us consider two positive numbers b and a_n with associated weights $n-1$ and 1 respectively.

Then $\frac{(n-1)b+1.a_n}{n} \geq (b^{n-1}.a_n)^{1/n}$, the equality occurs when $\frac{a_1+a_2+\cdots+a_{n-1}}{n-1} = a_n$

or, $\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n \geq a_n\left(\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}\right)^{n-1}$.

9. If a and b are positive rational numbers and $a > b$, prove that

$$a^{2a} < (a+b)^{a+b}(a-b)^{a-b}.$$

Let us consider two unequal positive numbers $\frac{1}{a+b}, \frac{1}{a-b}$ with associated positive rational weights $a+b$ and $a-b$ respectively.

Then $\frac{(a+b).\frac{1}{a+b}+(a-b).\frac{1}{a-b}}{2a} > \left[\left(\frac{1}{a+b}\right)^{a+b}\left(\frac{1}{a-b}\right)^{a-b}\right]^{1/2a}$

or, $\left(\frac{1}{a}\right)^{2a} > \frac{1}{(a+b)^{a+b}(a-b)^{a-b}}$

or, $a^{2a} < (a+b)^{a+b}(a-b)^{a-b}$.

10. If $a > 0$ but $\neq 1$ and x, y, z are rational numbers in descending order of magnitude, prove that $a^x(y-z) + a^y(z-x) + a^z(x-y) > 0$.

$x - y > 0$ and $y - z > 0$ since $x > y > z$.

Let us consider two unequal positive numbers a^x, a^z with associated positive rational weights $y-z$ and $x-y$ respectively.

Then $\frac{(y-z)a^x+(x-y)a^z}{x-z} > [a^{x(y-z)+z(x-y)}]^{1/(x-z)}$

or, $(y-z)a^x + (x-y)a^z > (x-z)a^y$

or, $a^x(y-z) + a^y(z-x) + a^z(x-y) > 0$.

Theorem 1.4.5. If a be a positive real number, not equal to 1, and x, y are positive rational numbers, then

$$\frac{a^x-1}{x} > \frac{a^y-1}{y} \quad \text{if } x > y.$$

Proof. Since $a > 0$ and $\neq 1, a^x > 0$ and $\neq 1$.

Let us consider two unequal positive numbers a^x and 1 with associated positive rational weights y and $x-y$ respectively.

Then $\frac{y.a^x+(x-y).1}{x} > [a^{xy}.1]^{1/x}$

or, $ya^x + x - y > xa^y$

or, $y(a^x - 1) > x(a^y - 1)$

or, $\frac{a^x-1}{x} > \frac{a^y-1}{y}$ since $x > 0, y > 0$.

Theorem 1.4.6. If a be a positive real number, not equal to 1, and m be a rational number, then $a^m - 1 >$ or $< m(a-1)$, according as m does not or does lie between 0 and 1.

Proof. **Case 1.** Let $m > 1$.

Since $a > 0$ and $\neq 1, a^m > 0$ and $\neq 1$.

Let us consider two unequal positive numbers a^m and 1 with associated positive rational weights 1 and $m-1$ respectively.

Then $\frac{1.a^m+(m-1).1}{m} > [a^m.1]^{1/m}$

or, $a^m + m - 1 > ma$

or, $a^m - 1 > m(a-1)$.

Case 2. Let $0 < m < 1$.

Let us consider two unequal positive numbers a and 1 with associated positive rational weights m and $1-m$ respectively.

Then $\frac{m.a+(1-m).1}{1} > a^m.1$

or, $m(a-1) > a^m - 1$

or, $a^m - 1 < m(a-1)$.

Case 3. Let $m < 0$. Then $1 - m > 1$.

Since $a > 0$ and $\neq 1, \frac{1}{a} > 0$ and $\neq 1$.

By Case 1, $\left(\frac{1}{a}\right)^{1-m} - 1 > (1-m)\left(\frac{1}{a} - 1\right)$

or, $\frac{a^m}{a} - 1 > (1-m)\frac{1-a}{a}$

or, $a^m - a > (1-m)(1-a)$, since $a > 0$

or, $a^m > 1 - m + am$

or, $a^m - 1 > m(a-1)$.

This completes the proof.

Deductions.

1. Let $x > -1$, but $\neq 0$ and m be a rational number. Then

$$(1+x)^m > \text{ or } < 1 + mx$$

according as m does not or does lie between 0 and 1.

[Take $a = 1 + x$. Then $a > 0$ but $\neq 1$.]

2. Let $x < 1$, but $\neq 0$ and m be a rational number. Then

$$(1-x)^m > \text{ or } < 1 - mx$$

according as m does not or does lie between 0 and 1.

[Take $a = 1 - x$. Then $a > 0$ but $\neq 1$.]

3. If a and b are unequal positive numbers and m is a rational numbe then $ma^{m-1}(a-b) >$ or $< a^m - b^m >$ or $< mb^{m-1}(a-b)$, according a m does not or does lie between 0 and 1.

[Take $k = \frac{a}{b}$. Then $k > 0$ but $\neq 1$. Again take $k = \frac{b}{a}$. Then $k >$ but $\neq 1$.]

Worked Examples (continued).

11. If n is a positive integer, prove that $(1+\frac{1}{n+1})^{n+1} > (1+\frac{1}{n})^n$.

Let us consider $n+1$ positive numbers $1+\frac{1}{n}, 1+\frac{1}{n}, \cdots, 1+\frac{1}{n}$ (n times) and 1. Applying A.M.>G.M., we have

$\frac{n.(1+\frac{1}{n})+1}{n+1} > (1+\frac{1}{n})^{\frac{n}{n+1}}$

or, $(\frac{n+2}{n+1})^{n+1} > (1+\frac{1}{n})^n$

or, $(1+\frac{1}{n+1})^{n+1} > (1+\frac{1}{n})^n$.

Alternative method

Let $a = 1+\frac{1}{n}$ and $m = \frac{n}{n+1}$. Then $a > 0$ but $\neq 1$ and m is a rational number and $0 < m < 1$. We have

$(1+\frac{1}{n})^{\frac{n}{n+1}} - 1 < \frac{n}{n+1}.(\frac{1}{n})$

or, $(1+\frac{1}{n})^{\frac{n}{n+1}} < 1 + \frac{1}{n+1}$

or, $(1+\frac{1}{n})^n < (1+\frac{1}{n+1})^{n+1}$

i.e., $(1+\frac{1}{n+1})^{n+1} > (1+\frac{1}{n})^n$.

Note. Let $u_n = (1+\frac{1}{n})^n$. Then $u_{n+1} > u_n$ for all $n \in \mathbf{N}$. This shows that the sequence $\{u_n\}$ is a strictly increasing sequence.

12. If n is a positive integer, prove that $(1+\frac{1}{n})^{n+1} > (1+\frac{1}{n+1})^{n+2}$.

Let us consider $n+2$ positive numbers $1-\frac{1}{n+1}, 1-\frac{1}{n+1}, \cdots, 1-\frac{1}{n+1}$ ($n+1$ times) and 1. Applying A.M.>G.M., we have

$\frac{(n+1).(1-\frac{1}{n+1})+1}{n+2} > (1-\frac{1}{n+1})^{\frac{n+1}{n+2}}$

or, $(\frac{n+1}{n+2})^{n+2} > (\frac{n}{n+1})^{n+1}$

or, $(\frac{n+1}{n})^{n+1} > (\frac{n+2}{n+1})^{n+2}$

or, $(1+\frac{1}{n})^{n+1} > (1+\frac{1}{n+1})^{n+2}$.

Alternative method

Let $a = 1-\frac{1}{n+1}$ and $m = \frac{n+1}{n+2}$. Then $a > 0$ but $\neq 1$ and m is a rational number and $0 < m < 1$. We have

$(1-\frac{1}{n+1})^{\frac{n+1}{n+2}} - 1 < \frac{n+1}{n+2}.\frac{1}{n+1}$

or, $(\frac{n}{n+1})^{\frac{n+1}{n+2}} < 1 - \frac{1}{n+2}$

or, $(\frac{n}{n+1})^{n+1} < (\frac{n+1}{n+2})^{n+2}$

or, $(1+\frac{1}{n})^{n+1} > (1+\frac{1}{n+1})^{n+2}$.

Note. Let $u_n = (1+\frac{1}{n})^{n+1}$. Then $u_{n+1} < u_n$ for all $n \in \mathbf{N}$. This shows that the sequence $\{u_n\}$ is a strictly decreasing sequence.

13. If a and b be positive real numbers and α be a rational number, prove that

$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b$ if $0 < \alpha < 1$,
$a^\alpha b^{1-\alpha} \geq \alpha a + (1-\alpha)b$ if $\alpha > 1$ or $\alpha < 0$.

Let $a \neq b, k = a/b$. Then $k > 0$ but $\neq 1$.

Therefore $k^\alpha - 1 >$ or $< \alpha(k-1)$, according as α does not or does lie between 0 and 1.

or, $(\frac{a}{b})^\alpha - 1 >$ or $< \alpha(\frac{a}{b} - 1)$

or, $b(\frac{a}{b})^\alpha - b >$ or $< \alpha(a-b)$

or, $a^\alpha b^{1-\alpha} >$ or $< \alpha a + (1-\alpha)b$, according as α does not or does lie between 0 and 1.

When $a = b$, $a^\alpha b^{1-\alpha} = \alpha a + (1-\alpha)b$.

14. Let $u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n$ be all positive real numbers and α be a rational number, prove that

$u_1^\alpha v_1^{1-\alpha} + u_2^\alpha v_2^{1-\alpha} + \cdots + u_n^\alpha v_n^{1-\alpha} \leq (u_1+u_2+\cdots+u_n)^\alpha (v_1+v_2+\cdots+v_n)^{1-\alpha}$, if $0 < \alpha < 1$;

$u_1^\alpha v_1^{1-\alpha} + u_2^\alpha v_2^{1-\alpha} + \cdots + u_n^\alpha v_n^{1-\alpha} \geq (u_1+u_2+\cdots+u_n)^\alpha (v_1+v_2+\cdots+v_n)^{1-\alpha}$, if $\alpha > 1$ or $\alpha < 0$.

The equality occurs when $u_i = \lambda v_i$ for some non-zero real λ.

Let $A = u_1 + u_2 + \cdots + u_n$, $B = v_1 + v_2 + \cdots + v_n$.

Then $\frac{u_1}{A} + \frac{u_2}{A} + \cdots + \frac{u_n}{A} = 1$, $\frac{v_1}{B} + \frac{v_2}{B} + \cdots + \frac{v_n}{B} = 1$.

Case 1. Let $0 < \alpha < 1$. We have

$(\frac{u_1}{A})^\alpha (\frac{v_1}{B})^{1-\alpha} \leq \frac{u_1}{A}\alpha + \frac{v_1}{B}(1-\alpha)$, the equality occurs if $u_1 : v_1 = A : B$

$(\frac{u_2}{A})^\alpha (\frac{v_2}{B})^{1-\alpha} \leq \frac{u_2}{A}\alpha + \frac{v_2}{B}(1-\alpha)$, the equality occurs if $u_2 : v_2 = A : B$

$(\frac{u_n}{A})^\alpha(\frac{v_n}{B})^{1-\alpha} \le \frac{u_n}{A}\alpha + \frac{v_n}{B}(1-\alpha)$, the equality occurs if $u_n : v_n = A : B$

Therefore $\frac{u_1^\alpha v_1^{1-\alpha}+u_2^\alpha v_2^{1-\alpha}+\cdots+u_n^\alpha v_n^{1-\alpha}}{A^\alpha B^{1-\alpha}} \le \alpha + (1-\alpha)$, the equality occurs when (u) and (v) are proportional

or, $u_1^\alpha v_1^{1-\alpha} + u_2^\alpha v_2^{1-\alpha} + \cdots + u_n^\alpha v_n^{1-\alpha} \le (u_1+u_2+\cdots+u_n)^\alpha (v_1+v_2+\cdots+v_n)^{1-\alpha}$, the equality occurs when (u) and (v) are proportional.

Case 2. Let $\alpha > 1$ or $\alpha < 0$.

Proceeding as in Case 1 the result is obtained.

15. If a and b be unequal positive real numbers and m be a rational number, prove that

$\frac{a^m+b^m}{2} >$ or $< (\frac{a+b}{2})^m$, according as m does not or does lie between 0 and 1.

Let $\frac{a+b}{2} = k$. Since $a \ne b, \frac{a}{k} \ne 1$ and $\frac{b}{k} \ne 1$.

Then $(\frac{a}{k})^m - 1 >$ or $< m(\frac{a}{k} - 1)$ according as m does not or does lie between 0 and 1;

and $(\frac{b}{k})^m - 1 >$ or $< m(\frac{b}{k} - 1)$ according as m does not or does lie between 0 and 1.

Therefore $\frac{a^m+b^m}{k^m} - 2 >$ or $< m(\frac{a+b}{k} - 2)$, according as m does not or does lie between 0 and 1

or, $\frac{a^m+b^m}{k^m} >$ or < 2

or, $\frac{a^m+b^m}{2} >$ or $< k^m$

or, $\frac{a^m+b^m}{2} >$ or $< (\frac{a+b}{2})^m$ according as m does not or does lie between 0 and 1.

Theorem 1.4.7. If $a_1, a_2, \ldots, a_n$ be n positive real numbers, not all equal, and m be a rational number, then

$$\frac{a_1^m+a_2^m+\cdots+a_n^m}{n} > \text{or} < \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m,$$

according as m does not or does lie between 0 and 1.

Proof. Let $k = \frac{a_1+a_2+\cdots+a_n}{n}$. Then $\frac{a_1}{k}, \frac{a_2}{k}, \cdots, \frac{a_n}{k}$ are all positive and not all of them are equal to 1.

Let $1 \le i \le n$. Then

$(\frac{a_i}{k})^m - 1 >$ or $< m(\frac{a_i}{k} - 1)$ according as m does not or does lie between 0 and 1. The inequality reduces to an equality if $\frac{a_i}{k} = 1$.

Considering n such relations for $i = 1, 2, \ldots, n$ and noting that all of them are not equalities, we have

$(\frac{a_1}{k})^m + (\frac{a_2}{k})^m + \cdots + (\frac{a_n}{k})^m - n >$ or $< m(\frac{a_1}{k} + \frac{a_2}{k} + \cdots + \frac{a_n}{k} - n)$, according as m does not or does lie between 0 and 1

or, $\frac{a_1^m+a_2^m+\cdots+a_n^m}{k^m} - n >$ or $< m(n-n)$

or, $\frac{a_1^m+a_2^m+\cdots+a_n^m}{k^m} >$ or $< n$

or, $\frac{a_1^m+a_2^m+\cdots+a_n^m}{n} >$ or $< (\frac{a_1+a_2+\cdots+a_n}{n})^m$, according as m does not or does lie between 0 and 1.

Theorem 1.4.8. If $a_1, a_2, \ldots, a_n$ be positive real numbers, not all equal, $p_1, p_2, \ldots, p_n$ be positive real numbers and m is rational, then

$\frac{p_1 a_1^m + p_2 a_2^m + \cdots + p_n a_n^m}{p_1+p_2+\cdots+p_n} >$ or $< (\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1+p_2+\cdots+p_n})^m$,

according as m does not or does lie between 0 and 1.

Proof. Let $q_i = \frac{p_i}{p_1+p_2+\cdots+p_n}$, $i = 1, 2, \ldots, n$. Then $q_1 + q_2 + \cdots + q_n = 1$.

Let $q_1 a_1 + q_2 a_2 + \cdots + q_n a_n = k$. Then

$\frac{a_1}{k}, \frac{a_2}{k}, \cdots, \frac{a_n}{k}$ are all positive and not all of them are equal to 1.

Let $1 \le i \le n$.

Then $(\frac{a_i}{k})^m - 1 >$ or $< m(\frac{a_i}{k} - 1)$, according as m does not or does lie between 0 and 1. The inequality reduces to an equality if $\frac{a_i}{k} = 1$.

Therefore $q_i(\frac{a_i}{k})^m - q_i >$ or $< m q_i(\frac{a_i}{k} - 1)$, since $q_i > 0$.

Considering n such relations for $i = 1, 2, \ldots, n$ and noting that all of them are not equalities, we have

$q_1(\frac{a_1}{k})^m + q_2(\frac{a_2}{k})^m + \cdots + q_n(\frac{a_n}{k})^m - (q_1 + q_2 + \cdots + q_n) >$ or

$< m[\frac{q_1 a_1 + q_2 a_2 + \cdots + q_n a_n}{k} - (q_1 + q_2 + \cdots + q_n)]$, according as m does not or does lie between 0 and 1.

or, $\frac{q_1 a_1^m + q_2 a_2^m + \cdots + q_n a_n^m}{k^m} - 1 >$ or $< m(1-1)$

or, $\frac{q_1 a_1^m + q_2 a_2^m + \cdots + q_n a_n^m}{k^m} >$ or < 1

or, $q_1 a_1^m + q_2 a_2^m + \cdots + q_n a_n^m >$ or $< (q_1 a_1 + q_2 a_2 + \cdots + q_n a_n)^m$

or, $\frac{p_1 a_1^m + p_2 a_2^m + \cdots + p_n a_n^m}{p_1+p_2+\cdots+p_n} >$ or $< (\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1+p_2+\cdots+p_n})^m$, according as m does not or does lie between 0 and 1.

Corollary. If $a_1, a_2, \ldots, a_n$ be positive real numbers, not all equal, and $q_1, q_2, \ldots, q_n$ be positive real numbers such that $q_1 + q_2 + \cdots + q_n = 1$ and m is rational, then $q_1 a_1^m + q_2 a_2^m + \cdots + q_n a_n^m >$ or $< (q_1 a_1 + q_2 a_2 + \cdots + q_n a_n)^m$ according as m does not or does lie between 0 and 1.

Worked Examples (continued).

16. If a, b, c be unequal positive real numbers such that the sum of any two is greater than the third, prove that

$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} > \frac{9}{a+b+c}$.

$b+c-a > 0, c+a-b > 0, a+b-c > 0$ and they are unequal.

Then $\frac{(b+c-a)^{-1}+(c+a-b)^{-1}+(a+b-c)^{-1}}{3} > \left[\frac{(b+c-a)+(c+a-b)+(a+b-c)}{3}\right]^{-1}$

or, $\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} > \frac{9}{a+b+c}$.

17. If a, b are positive real numbers and $a+b=4$, prove that

$(a+\frac{1}{a})^2 + (b+\frac{1}{b})^2 \geq \frac{25}{2}$.

$(a+\frac{1}{a})^2 + (b+\frac{1}{b})^2 = (a^2+b^2) + (a^{-2}+b^{-2}) + 4$.

We have $\frac{a^m+b^m}{2} >$ or $< (\frac{a+b}{2})^m$ according as m does not or does lie between 0 and 1.

Let $m=2$. Then $\frac{a^2+b^2}{2} \geq (\frac{a+b}{2})^2$, the equality occurs when $a=b$

or, $a^2+b^2 \geq 8$.

Let $m=-2$. Then $\frac{a^{-2}+b^{-2}}{2} \geq (\frac{a+b}{2})^{-2}$, the equality occurs when $a=b$

or, $a^{-2}+b^{-2} \geq \frac{1}{2}$.

Therefore $(a+\frac{1}{a})^2 + (b+\frac{1}{b})^2 \geq \frac{25}{2}$, the equality occurs when $a=b$.

18. If a, b, c be three positive real numbers in harmonic progression and n be a positive integer greater than 1, prove that

$$a^n + c^n > 2b^n.$$

Since a, c are unequal positive real numbers and $n > 1$,

$\frac{a^n+c^n}{2} > (\frac{a+c}{2})^n$ ... (i)

Since a, b, c are in H.P., $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$.

But $\frac{a^{-1}+c^{-1}}{2} > (\frac{a+c}{2})^{-1}$, since a, c are unequal positive real numbers.

Therefore $\frac{1}{b} > (\frac{a+c}{2})^{-1}$, i.e., $\frac{a+c}{2} > b$.

Therefore from (i) $a^n + c^n > 2b^n$.

19. If $a_1, a_2, \ldots, a_n$ are unequal positive real numbers in A.P., show that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} > \frac{2n}{a_1+a_n}$.

We have $\frac{a_1^{-1}+a_2^{-1}+\cdots+a_n^{-1}}{n} > (\frac{a_1+a_2+\cdots+a_n}{n})^{-1}$

or, $a_1^{-1} + a_2^{-1} + \cdots + a_n^{-1} > n \cdot \frac{n}{a_1+a_2+\cdots+a_n}$.

Since $a_1, a_2, \ldots, a_n$ are in A.P., $a_1 + a_2 + \cdots + a_n = \frac{(a_1+a_n)n}{2}$.

Therefore $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} > \frac{2n}{a_1+a_n}$.

1.5. Application to the problems of maxima and minima.

Let $x_1, x_2, \ldots, x_n$ be n positive real numbers. Then $(\frac{x_1+x_2+\cdots+x_n}{n})^n \geq x_1 x_2 \ldots x_n$, the equality occurs when $x_1 = x_2 = \ldots = x_n$.

1. Let $x_1, x_2, \ldots, x_n$ be n *positive real variables* such that their sum $x_1 + x_2 + \cdots + x_n$ is a constant, k. Then

$$(\tfrac{k}{n})^n \geq x_1 x_2 \ldots x_n.$$

Therefore the *maximum* value of $x_1 x_2 \ldots x_n$ occurs when $x_1 = x_2 = \ldots = x_n = \frac{k}{n}$ and the maximum value is $(\frac{k}{n})^n$.

2. Let $x_1, x_2, \ldots, x_n$ be n *positive real variables* such that their product $x_1 x_2 \ldots x_n$ is a constant, k. Then $\frac{x_1+x_2+\cdots+x_n}{n} \geq k^{1/n}$.

Therefore the *minimum* value of $x_1 + x_2 + \cdots + x_n$ occurs when $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$ and the minimum value is $nk^{1/n}$.

Let $x_1, x_2, \ldots, x_n$ be n positive real numbers and m is a rational number other than 0 and 1. Then

$$\frac{x_1{}^m + x_2{}^m + \cdots + x_n{}^m}{n} > \text{ or } < (\frac{x_1+x_2+\cdots+x_n}{n})^m,$$

according as m does not or does lie betwen 0 and 1. The inequality reduces to an equality if $x_1 = x_2 = \cdots = x_n$.

1. Let $x_1, x_2, \ldots, x_n$ be n *positive real variables* such that $x_1 + x_2 + \ldots + x_n = k$, a constant, and m is a rational number other than 0 and 1. Then

$$\frac{x_1{}^m + x_2{}^m + \cdots + x_n{}^m}{n} \geq (\tfrac{k}{n})^m \text{ when } m > 1, \text{ or } m < 0;$$

$$\frac{x_1{}^m + x_2{}^m + \cdots + x_n{}^m}{n} \leq (\tfrac{k}{n})^m \text{ when } 0 < m < 1.$$

Therefore when $m > 1$ or $m < 0$, the minimum value of $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$ occurs when $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$ and the minimum value is $n(\frac{k}{n})^m$.

And when $0 < m < 1$, the maximum value of $x_1{}^m + x_2{}^m + \cdots + x_n{}^m$ occurs when $x_1 = x_2 = \cdots = x_n = \frac{k}{n}$ and the maximum value is $n(\frac{k}{n})^m$.

2. Let $x_1, x_2, \ldots, x_n$ be n *positive real variables* such that $x_1{}^m + x_2{}^m + \cdots + x_n{}^m = k$, a constant, and m is a rational number other than 0 and 1. Then

$$\frac{k}{n} \geq (\tfrac{x_1+x_2+\cdots+x_n}{n})^m \text{ when } m > 1;$$

$$\frac{k}{n} \leq (\tfrac{x_1+x_2+\cdots+x_n}{n})^m \text{ when } 0 < m < 1;$$

$$\frac{k}{n} \geq (\tfrac{x_1+x_2+\cdots+x_n}{n})^m \text{ when } m < 0.$$

cla 3

Hence $(\frac{k}{n})^{1/m} \geq (\frac{x_1+x_2+\cdots+x_n}{n})$ when $m > 1$;

$(\frac{k}{n})^{1/m} \leq (\frac{x_1+x_2+\cdots+x_n}{n})$ when $0 < m < 1$;

and $(\frac{k}{n})^{1/m} \leq (\frac{x_1+x_2+\cdots+x_n}{n})$ when $m < 0$.

Therefore when $m > 1$, the maximum value of $x_1 + x_2 + \cdots + x_n$ occurs when $x_1 = x_2 = \cdots = x_n = (\frac{k}{n})^{1/m}$ and the maximum value is $n(\frac{k}{n})^{1/m}$.

And when $0 < m < 1$, or when $m < 0$, the minimum value of $x_1 + x_2 + \cdots + x_n$ occurs when $x_1 = x_2 = \cdots = x_n = (\frac{k}{n})^{1/m}$ and the minimum value is $n(\frac{k}{n})^{1/m}$.

Theorem 1.5.1. Let $\alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2 \ldots, \beta_n$ be all positive rational numbers and let $x_1, x_2, \ldots, x_n$ be n positive real variables such that $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n$ is a constant. Then the maximum value of ${x_1}^{\alpha_1} {x_2}^{\alpha_2} \ldots {x_n}^{\alpha_n}$ occurs when

$$\frac{\beta_1 x_1}{\alpha_1} = \frac{\beta_2 x_2}{\alpha_2} = \cdots = \frac{\beta_n x_n}{\alpha_n}.$$

Proof. Let us consider n positive numbers $\frac{\beta_1 x_1}{\alpha_1}, \frac{\beta_2 x_2}{\alpha_2}, \ldots, \frac{\beta_n x_n}{\alpha_n}$ with associated positive rational weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively. Then

$$\frac{\alpha_1 . \frac{\beta_1 x_1}{\alpha_1} + \alpha_2 . \frac{\beta_2 x_2}{\alpha_2} + \cdots + \alpha_n . \frac{\beta_n x_n}{\alpha_n}}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \geq \left[\left(\frac{\beta_1 x_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{\beta_2 x_2}{\alpha_2}\right)^{\alpha_2} \ldots \left(\frac{\beta_n x_n}{\alpha_n}\right)^{\alpha_n}\right]^{\frac{1}{\alpha_1+\alpha_2+\cdots+\alpha_n}}$$

the equality occurs when $\frac{\beta_1 x_1}{\alpha_1} = \frac{\beta_2 x_2}{\alpha_2} = \cdots = \frac{\beta_n x_n}{\alpha_n}$.

Hence $(\frac{k}{\Sigma \alpha_i})^{\Sigma \alpha_i} \geq (\frac{\beta_1}{\alpha_1})^{\alpha_1} (\frac{\beta_2}{\alpha_2})^{\alpha_2} \ldots (\frac{\beta_n}{\alpha_n})^{\alpha_n} . {x_1}^{\alpha_1} {x_2}^{\alpha_2} \ldots {x_n}^{\alpha_n}$, where $k = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n$.

Therefore the maximum value of ${x_1}^{\alpha_1} {x_2}^{\alpha_2} \ldots {x_n}^{\alpha_n}$ occurs when $\frac{\beta_1 x_1}{\alpha_1} = \frac{\beta_2 x_2}{\alpha_2} = \cdots = \frac{\beta_n x_n}{\alpha_n}$ and the maximum value is

$$\left(\frac{\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}\right)^{\Sigma \alpha_i} . \left(\frac{\alpha_1}{\beta_1}\right)^{\alpha_1} \left(\frac{\alpha_2}{\beta_2}\right)^{\alpha_2} \ldots \left(\frac{\alpha_n}{\beta_n}\right)^{\alpha_n}.$$

Theorem 1.5.2. Let $\alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2, \ldots, \beta_n$ be all positive rational numbers and let $x_1, x_2, \ldots, x_n$ be n positive real variables such that ${x_1}^{\alpha_1} {x_2}^{\alpha_2} \ldots {x_n}^{\alpha_n}$ is a constant. Then the minimum value of $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n$ occurs when $\frac{\beta_1 x_1}{\alpha_1} = \frac{\beta_2 x_2}{\alpha_2} = \ldots = \frac{\beta_n x_n}{\alpha_n}$.

Proof. Proceed as in the previous theorem.

Worked Examples.

1. Find the greatest value of $a^2 b^3$ where a and b are positive real numbers satisfying $a + b = 10$. Determine the values of a, b for which the greatest value is attained.

Let us consider two positive numbers $\frac{a}{2}, \frac{b}{3}$ with associated weights 2 and 3 respectively. Then

$\frac{2.\frac{a}{2} + 3.\frac{b}{3}}{2+3} \geq [(\frac{a}{2})^2 (\frac{b}{3})^3]^{1/5}$, the equality occurs when $\frac{a}{2} = \frac{b}{3}$

That is, $(\frac{10}{5})^5 \geq \frac{a^2 b^3}{108}$, or, $a^2 b^3 \leq 3456$.

Therefore the greatest value is 3456 and this is attained when $\frac{a}{2} = \frac{b}{3} = \frac{a+b}{5} = 2$, i.e., when $a = 4, b = 6$.

2. Find the maximum value of $(x+2)^5 (7-x)^4$ when $-2 < x < 7$.

$x + 2 > 0, 7 - x > 0$ and $(x + 2) + (7 - x) = 9$.

Let us consider two positive numbers $\frac{x+2}{5}, \frac{7-x}{4}$ with associated weights 5 and 4 respectively. Then

$\frac{5.\frac{x+2}{5} + 4.\frac{7-x}{4}}{9} \geq [(\frac{x+2}{5})^5 (\frac{7-x}{4})^4]^{1/9}$, the equality occurs when $\frac{x+2}{5} = \frac{7-x}{4}$

or, $5^5 4^4 \geq (x+2)^5 (7-x)^4$.

Therefore the maximum value is $5^5 4^4$ and the maximum is attained when $\frac{x+2}{5} = \frac{7-x}{4} = \frac{(x+2)+(7-x)}{5+4} = 1$, i.e., when $x = 3$.

3. Find the greatest value of $(2x+1)^3 (y+2)^2$ when $x + y = 3$ and $-\frac{1}{2} < x < 5$.

Here $2x + 1 > 0$ and $y + 2 = 5 - x > 0$.

Let us consider positive numbers $\frac{2x+1}{3}, y+2$ with associated weights 3 and 2 respectively.Then

$\frac{3.(\frac{2x+1}{3}) + 2.(y+2)}{5} \geq [(\frac{2x+1}{3})^3 (y+2)^2]^{1/5}$, the equality when $\frac{2x+1}{3} = y+2$.

or, $27(\frac{11}{5})^5 \geq (2x+1)^3 (y+2)^2$.

Therefore the greatest value is $27(\frac{11}{5})^5$ and this is attained when $\frac{2x+1}{3} = \frac{y+2}{1} = \frac{(2x+1)+2(y+2)}{5} = \frac{11}{5}$, i.e., when $x = \frac{14}{5}, y = \frac{1}{5}$.

4. Find the minimum value of $3x + 2y$ when x, y are positive real numbers satisfying the condition $x^2 y^3 = 48$.

Let us consider positive numbers $\frac{3x}{2}, \frac{2y}{3}$ with associated weights 2 and 3 respectively. Then

$\frac{2(\frac{3x}{2}) + 3(\frac{2y}{3})}{5} \geq [(\frac{3x}{2})^2 (\frac{2y}{3})^3]^{1/5}$, the equality occurs when $\frac{3x}{2} = \frac{2y}{3}$.

or, $3x + 2y \geq 10$.

Therefore the minimum value is 10 and this is attained when $\frac{3x}{2} = \frac{2y}{3} = [(\frac{3x}{2})^2 . (\frac{2y}{3})^3]^{1/5} = 2$, i.e., when $x = \frac{4}{3}, y = 3$.

1.6. Standard inequalities.

1.6.1. Holder's inequality.

Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n; \ldots \ldots; l_1, l_2, \ldots, l_n$ be all positive real numbers and $\alpha, \beta, \ldots, \lambda$ be positive rational numbers such that $\alpha + \beta + \cdots + \lambda = 1$. Then

$a_1^\alpha b_1^\beta \ldots l_1^\lambda + a_2^\alpha b_2^\beta \ldots l_2^\lambda + \cdots + a_n^\alpha b_n^\beta \ldots l_n^\lambda \leq (a_1 + a_2 + \cdots + a_n)^\alpha (b_1 + b_2 + \cdots + b_n)^\beta \ldots (l_1 + l_2 + \cdots + l_n)^\lambda$,
the equality occurs when $(a), (b), \cdots, (l)$ are proportional.

Proof. Let $A = a_1 + a_2 + \cdots + a_n, B = b_1 + b_2 + \cdots + b_n, \ldots, L = l_1 + l_2 + \cdots + l_n$.

Let us consider positive numbers $\frac{a_1}{A}, \frac{b_1}{B}, \ldots, \frac{l_1}{L}$ with associated positive rational weights $\alpha, \beta, \ldots, \lambda$ respectively.

Then $(\frac{a_1}{A})^\alpha (\frac{b_1}{B})^\beta \ldots (\frac{l_1}{L})^\lambda \leq \alpha(\frac{a_1}{A}) + \beta(\frac{b_1}{B}) + \cdots + \lambda(\frac{l_1}{L})$, the equality occurs when $a_1 : b_1 : \cdots : l_1 = A : B : \cdots : L$.

Similarly, $(\frac{a_2}{A})^\alpha (\frac{b_2}{B})^\beta \ldots (\frac{l_2}{L})^\lambda \leq \alpha(\frac{a_2}{A}) + \beta(\frac{b_2}{B}) + \cdots + \lambda(\frac{l_2}{L})$, the equality occurs when $a_2 : b_2 : \cdots : l_2 = A : B : \cdots : L$.

... ...

$(\frac{a_n}{A})^\alpha (\frac{b_n}{B})^\beta \ldots (\frac{l_n}{L})^\lambda \leq \alpha(\frac{a_n}{A}) + \beta(\frac{b_n}{B}) + \cdots + \lambda(\frac{l_n}{L})$, the equality occurs when $a_n : b_n : \cdots : l_n = A : B : \cdots : L$.

Therefore $\frac{a_1^\alpha b_1^\beta \ldots l_1^\lambda + a_2^\alpha b_2^\beta \ldots l_2^\lambda + \cdots + a_n^\alpha b_n^\beta \ldots l_n^\lambda}{A^\alpha B^\beta \ldots L^\lambda} \leq \alpha + \beta + \cdots + \lambda$, the equality occurs when $(a), (b), \ldots, (l)$ are proportional.

or, $a_1^\alpha b_1^\beta \ldots l_1^\lambda + a_2^\alpha b_2^\beta \ldots l_2^\lambda + \cdots + a_n^\alpha b_n^\beta \ldots l_n^\lambda \leq A^\alpha B^\beta \ldots L^\lambda$, since $\alpha + \beta + \ldots + \lambda = 1$.

or, $a_1^\alpha b_1^\beta \ldots l_1^\lambda + a_2^\alpha b_2^\beta \ldots l_2^\lambda + \cdots + a_n^\alpha b_n^\beta \ldots l_n^\lambda \leq (a_1 + a_2 + \cdots + a_n)^\alpha (b_1 + b_2 + \cdots + b_n)^\beta \ldots (l_1 + l_2 + \cdots + l_n)^\lambda$, the equality occurs when $(a), (b), \ldots, (l)$ are proportional.

This completes the proof.

Particular cases.

(i) Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, \beta_n$ be positive real numbers and λ, μ be positive rational numbers such that $\lambda + \mu = 1$.

Then $a_1^\lambda b_1^\mu + a_2^\lambda b_2^\mu + \cdots + a_n^\lambda b_n^\mu \leq (a_1 + a_2 + \cdots + a_n)^\lambda (b_1 + b_2 + \cdots + b_n)^\mu$, the equality occurs when (a) and (b) are proportional.

(ii) Let $p > 1, q > 1$ be rational numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.

Let us consider positive real numbers $a_1^p, a_2^p, \ldots, a_n^p$; $b_1^q, b_2^q, \ldots, b_n^q$.

Then $(a_1^p)^{1/p}(b_1)^{1/q} + (a_2^p)^{1/p}(b_2^q)^{1/q} + \cdots + (a_n^p)^{1/p}(b_n^q)^{1/q} \leq (a_1^p + a_2^p + \cdots + a_n^p)^{1/p}(b_1^q + b_2^q + \cdots + b_n^q)^{1/q}$, the equality occurs when (a) and (b) are proportional.

or, $(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \leq (a_1^p + a_2^p + \cdots + a_n^p)^{1/p}(b_1^q + b_2^q + \cdots + b_n^q)^{1/q}$, the equality occurs when (a) and (b) are proportional.

In particular, if $p = 2, q = 2$ then $(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$, the equality occurs when (a) and (b) are proportional. This is Cauchy-Schwarz inequality.

1.6.2. Jensen's inequality.

Let $a_1, a_2, \ldots, a_n$ be n positive real numbers and r, s are positive rational numbers. Then

$(a_1^r + a_2^r + \cdots + a_n^r)^{1/r} > (a_1^s + a_2^s + \cdots + a_n^s)^{1/s}$, if $r < s$.

In symbols, $G_r(a) > G_s(a)$ if $0 < r < s$, where $G_r(a) = (a_1^r + a_2^r + \ldots + a_n^r)^{1/r}$.

Proof. Let $A = a_1^r + a_2^r + \cdots + a_n^r$.

Then $\frac{a_1^r}{A} + \frac{a_2^r}{A} + \cdots + \frac{a_n^r}{A} = 1$ and $0 < \frac{a_i^r}{A} < 1$ for $i = 1, 2, \ldots, n$.

$\frac{s}{r} > 1$ and therefore $(\frac{a_i^r}{A})^{s/r} < \frac{a_i^r}{A}$ for $i = 1, 2, \ldots, n$

or, $a_i^s < A^{s/r-1} . a_i^r$ for $i = 1, 2, \ldots, n$

Therefore $(a_1^s + a_2^s + \cdots + a_n^s) < A^{s/r-1}(a_1^r + a_2^r + \cdots + a_n^r)$

or, $(a_1^s + a_2^s + \cdots + a_n^s) < (a_1^r + a_2^r + \cdots + a_n^r)^{s/r}$

or, $(a_1^s + a_2^s + \cdots + a_n^s)^{1/s} < (a_1^r + a_2^r + \cdots + a_n^r)^{1/r}$.

This completes the proof.

Corollary. $a_1^r + a_2^r + \cdots + a_n^r >$ or $< (a_1 + a_2 + \cdots + a_n)^r$ according as $0 < r < 1$ or $r > 1$.

If $0 < r < 1$, then by the theorem, we have $G_r(a) > G_1(a)$ and if $r > 1, G_1(a) > G_r(a)$.

1.6.3. Minkowski's inequality.

Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be all positive real numbers and r is a positive rational number. Then

$(a_1^r + a_2^r + \cdots + a_n^r)^{1/r} + (b_1^r + b_2^r + \cdots + b_n^r)^{1/r}$
$\geq [(a_1 + b_1)^r + (a_2 + b_2)^r + \cdots + (a_n + b_n)^r]^{1/r}$, when $r > 1$;

$(a_1^r + a_2^r + \cdots + a_n^r)^{1/r} + (b_1^r + b_2^r + \cdots + b_n^r)^{1/r}$
$\leq [(a_1 + b_1)^r + (a_2 + b_2)^r + \cdots + (a_n + b_n)^r]^{1/r}$, when $0 < r < 1$; the equality occurs when (a) and (b) are proportional.

Proof. Let $A^r = (a_1{}^r + a_2{}^r + \cdots + a_n{}^r)$, $B^r = b_1{}^r + b_2{}^r + \cdots + b_n{}^r$.

Let us consider positive numbers $\frac{a_1}{A}, \frac{b_1}{B}$. By theorem 1.4.8,

$\frac{A(\frac{a_1}{A})^r + B(\frac{b_1}{B})^r}{A+B} \geq \left(\frac{A\frac{a_1}{A} + B\frac{b_1}{B}}{A+B}\right)^r$, when $r > 1$. The equality occurs when $a_1 : b_1 = A : B$

or, $\frac{A(\frac{a_1}{A})^r + B(\frac{b_1}{B})^r}{A+B} \geq \left(\frac{a_1+b_1}{A+B}\right)^r$.

Similarly, $\frac{A(\frac{a_2}{A})^r + B(\frac{b_2}{B})^r}{A+B} \geq \left(\frac{a_2+b_2}{A+B}\right)^r$, the equality occurs when $a_2 : b_2 = A : B$

... ...

$\frac{A(\frac{a_n}{A})^r + B(\frac{b_n}{B})^r}{A+B} \geq \left(\frac{a_n+b_n}{A+B}\right)^r$, the equality occurs when $a_n : b_n = A : B$

Therefore, $\frac{A+B}{A+B} \geq \frac{(a_1+b_1)^r + (a_2+b_2)^r + \cdots + (a_n+b_n)^r}{(A+B)^r}$, the equality occurs when (a) and (b) are proportional

or, $(A+B)^r \geq (a_1+b_1)^r + (a_2+b_2)^r + \cdots + (a_n+b_n)^r$

or, $(a_1{}^r + a_2{}^r + \cdots + a_n{}^r)^{1/r} + (b_1{}^r + b_2{}^r + \cdots + b_n{}^r)^{1/r} \geq [(a_1 + b_1)^r + (a_2 + b_2)^r + \cdots + (a_n + b_n)^r]^{1/r}$.

The inequality sign is reversed when $0 < r < 1$.

This completes the proof.

Note. The theorem can be generalised to a finite number of sets.

Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n; \ \ldots \ ; l_1, l_2, \ldots, l_n$ be all positive real numbers and r is a positive rational number. Then

$(a_1{}^r + a_2{}^r + \cdots + a_n{}^r)^{1/r} + (b_1{}^r + b_2{}^r + \cdots + b_n{}^r)^{1/r} + \cdots + (l_1{}^r + l_2{}^r + \cdots + l_n{}^r)^{1/r} \geq [(a_1 + b_1 + \cdots + l_1)^r + (a_2 + b_2 + \cdots + l_2)^r + \cdots + (a_n + b_n + \cdots + l_n)^r]^{1/r}$, when $r > 1$;

$(a_1{}^r + a_2{}^r + \cdots + a_n{}^r)^{1/r} + (b_1{}^r + b_2{}^r + \cdots + b_n{}^r)^{1/r} + \cdots + (l_1{}^r + l_2{}^r + \cdots + l_n{}^r)^{1/r} \leq [(a_1 + b_1 + \cdots + l_1)^r + (a_2 + b_2 + \cdots + l_2)^r + \cdots + (a_n + b_n + \cdots + l_n)^r]^{1/r}$, when $0 < r < 1$.

The equality occurs when $(a), (b), \ldots, (l)$ are proportional.

Worked Examples.

1. If $a_1, b_1; a_2, b_2; a_3, b_3$ be positive real numbers, prove that $(a_1a_2a_3 + b_1b_2b_3)^3 \leq (a_1{}^3 + b_1{}^3)(a_2{}^3 + b_2{}^3)(a_3{}^3 + b_3{}^3)$.

By Holder's inequality,

$a_1{}^\alpha a_2{}^\beta a_3{}^\gamma + b_1{}^\alpha b_2{}^\beta b_3{}^\gamma \leq (a_1+b_1)^\alpha (a_2+b_2)^\beta (a_3+b_3)^\gamma$, where α, β, γ are positive rational numbers such that $\alpha + \beta + \gamma = 1$

Let us consider $a_1{}^3, b_1{}^3; a_2{}^3, b_2{}^3; a_3{}^3, b_3{}^3$ and let $\alpha = \beta = \gamma = \frac{1}{3}$.

Then $a_1a_2a_3 + b_1b_2b_3 \leq (a_1{}^3 + b_1{}^3)^{1/3}(a_2{}^3 + b_2{}^3)^{1/3}(a_3{}^3 + b_3{}^3)^{1/3}$

or, $(a_1a_2a_3 + b_1b_2b_3)^3 \leq (a_1{}^3 + b_1{}^3)(a_2{}^3 + b_2{}^3)(a_3{}^3 + b_3{}^3)$.

2. If a, b, c, d be all positive real numbers, prove that $(a^4 + b^4 + c^4 + d^4)^3 < (a^3 + b^3 + c^3 + d^3)^4$.

By Jensen's inequality, $G_r(a) < G_s(a)$ if $r > s > 0$.

Taking $r = 4$ and $s = 3$, $(a^4 + b^4 + c^4 + d^4)^{1/4} < (a^3 + b^3 + c^3 + d^3)^{1/3}$

or, $(a^4 + b^4 + c^4 + d^4)^3 < (a^3 + b^3 + c^3 + d^3)^4$.

3. In a triangle ABC, $a^3 + b^3 = c^3$. Prove that C is an acute angle.

By Jensen's inequality, $(a^3 + b^3)^{1/3} < (a^2 + b^2)^{1/2}$

or, $c^2 < a^2 + b^2$.

Therefore C is an acute angle.

4. If a, b, c be all positive real numbers and $a^3 + b^3 + c^3 = 1$ show that $(a+3)^3 + (b+4)^3 + (c+5)^3 \leq 343$. Discuss the case of equality.

By Minkowski's inequality,

$(a^3+b^3+c^3)^{1/3} + (3^3+4^3+5^3)^{1/3} \geq [(a+3)^3 + (b+4)^3 + (c+5)^3]^{1/3}$, since here $r = 3 > 1$.

The equality occurs when $\frac{a}{3} = \frac{b}{4} = \frac{c}{5}$.

or, $[1 + (216)^{1/3}]^3 \geq (a+3)^3 + (b+4)^3 + (c+5)^3$

or, $(a+3)^3 + (b+4)^3 + (c+5)^3 \leq 343$.

The equality occurs when $\frac{a}{3} = \frac{b}{4} = \frac{c}{5}$, i.e., when $a = \frac{3}{6}, b = \frac{4}{6}, c = \frac{5}{6}$, since $a^3 + b^3 + c^3 = 1$.

5. If $a^{1/3} + b^{1/3} + c^{1/3} = (127)^{1/3}$, show that $(a+1)^{1/3} + (b+8)^{1/3} + (c+27)^{1/3} \geq 7$. Discuss the case of equality.

By Minkowski's inequality,

$(a^{1/3} + b^{1/3} + c^{1/3})^3 + (1^{1/3} + 8^{1/3} + 27^{1/3})^3 \leq [(a+1)^{1/3} + (b+8)^{1/3} + (c+27)^{1/3})]^3$, since here $r = \frac{1}{3}$.

The equality occurs when $\frac{a}{1} = \frac{b}{8} = \frac{c}{27}$.

or, $127 + 216 \leq [(a+1)^{1/3} + (b+8)^{1/3} + (c+27)^{1/3}]^3$

or, $(a+1)^{1/3} + (b+8)^{1/3} + (c+27)^{1/3} \geq (343)^{1/3}$

or, $(a+1)^{1/3} + (b+8)^{1/3} + (c+27)^{1/3} \geq 7$.

The equality occurs when $\frac{a}{1} = \frac{b}{8} = \frac{c}{27}$, i.e., when $a = \frac{127}{216}, b = \frac{127}{27}, c = \frac{127}{8}$, since $a^{1/3} + b^{1/3} + c^{1/3} = (127)^{1/3}$.

Exercises 1B

1. If a, b, c be positive real numbers, prove that

(i) $a^4 + b^4 + c^4 \geq abc(a+b+c)$,

(ii) $\left(\frac{a+b+c}{3}\right)^3 \geq a\left(\frac{b+c}{2}\right)^2$,

(iii) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$,

(iv) $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$,

(v) $\frac{9}{a+b+c} \leq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$,

(vi) $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} > \frac{3}{2}$, unless $a = b = c$

(vii) $(ab + bc + ca)(ab^{-1} + bc^{-1} + ca^{-1}) \geq (a+b+c)^2$.

2. If a, b, c be all positive real numbers and $a + b + c = 1$, prove that

(i) $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} \geq \frac{9}{2}$,

(ii) $(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 + (c + \frac{1}{c})^2 \geq 33\frac{1}{3}$,

(iii) $(1+a)(1+b)(1+c) \geq 8(1-a)(1-b)(1-c)$.

[**Hint.** (iii) $\frac{1+a}{2} = \frac{(a+b)+(c+a)}{2} \geq \sqrt{(a+b)(c+a)}$, i.e., $\geq \sqrt{(1-c)(1-b)}$.]

3. If a, b, c be three positive real numbers such that the sum of any two is greater than the third, prove that

(i) $\frac{ab+bc+ca}{a^2+b^2+c^2} > \frac{1}{2}$,

(ii) $(a+b+c)^3 \geq 27(a+b-c)(b+c-a)(c+a-b)$,

(iii) $abc \geq 8(s-a)(s-b)(s-c)$, where $2s = a+b+c$,

(iv) $\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3$.

[**Hint.** (iv) Let $b+c-a = x, c+a-b = y, a+b-c = z$. Then $x > 0, y > 0, z > 0$ and $a = \frac{1}{2}(y+z), b = \frac{1}{2}(z+x), c = \frac{1}{2}(x+y)$].

4. a, b, c, d **are positive real numbers. Prove that**

(i) $a^6 + b^6 + c^6 + d^6 \geq abcd(a^2 + b^2 + c^2 + d^2)$,

(ii) $\left(\frac{a+b+c+d}{4}\right)^4 \geq ab\left(\frac{c+d}{2}\right)^2$,

(iii) $(1+a^4)(1+b^4)(1+c^4)(1+d^4) \geq (1+abcd)^4$,

(iv) $4(a^4 + b^4 + c^4 + d^4) \geq (a+b+c+d)(a^3 + b^3 + c^3 + d^3) \geq 16abcd$,

(v) $\frac{a^2+b^2+c^2}{a+b+c} + \frac{b^2+c^2+d^2}{b+c+d} + \frac{c^2+d^2+a^2}{c+d+a} + \frac{d^2+a^2+b^2}{d+a+b} \geq a+b+c+d$.

5. If a, b, c, d **be all positive real numbers and** $s = a+b+c+d$, **prove that**

(i) $81abcd \leq (s-a)(s-b)(s-c)(s-d) \leq \frac{81}{256}s^4$,

(ii) $\frac{16}{s} \leq \frac{3}{s-a} + \frac{3}{s-b} + \frac{3}{s-c} + \frac{3}{s-d} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$.

6. If n **be a positive integer** > 1, **prove that**

(i) $(\frac{n+1}{2})^n > n!$,

(ii) $n^n > 1.3.5 \cdots (2n-1)$,

(iii) $2^n > 1 + n.2^{(n-1)/2}$,

(iv) $\frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots 2n} > \frac{1}{2\sqrt{n}}$.

[**Hint.** (iv) $\frac{n+(n-1)}{2} > \sqrt{n(n-1)}$, i.e., $\frac{2n-1}{2n} > \sqrt{\frac{n-1}{n}}$.]

7. If n **be a positive integer, prove that**

(i) $\frac{1}{2\sqrt{n+1}} < \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots 2n} < \frac{1}{\sqrt{2n+1}}$,

(ii) $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$,

(iii) $(1 - \frac{1}{n+1})^{n+1} > (1 - \frac{1}{n})^n$.

[**Hint.** (iii) Take $a = 1 - \frac{1}{n+1}$ and $m = \frac{n+1}{n}$. Then $a^m - 1 > m(a-1)$.]

8. If $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, **prove that**

(i) $s_n > \frac{2n}{n+1}$, if $n > 1$

(ii) $n + s_n > n(n+1)^{1/n}$, if $n > 1$

(iii) $\left(\frac{n-s_n}{n-1}\right)^{n-1} > \frac{1}{n}$, if $n > 2$.

[**Hint.** (ii) Consider n positive numbers $1+1, 1+\frac{1}{2}, \ldots, 1+\frac{1}{n}$. Apply A.M $\geq$ G.M. Strict inequality occurs if $n > 1$.

(iii) Consider $n-1$ positive numbers $1-\frac{1}{2}, 1-\frac{1}{3}, \ldots, 1-\frac{1}{n}$. Apply A.M $\geq$ G.M. Strict inequality occurs if $n - 1 > 1$.]

9. (i) If $a_1, a_2, \ldots, a_n$ be all positive real numbers and $a_1a_2 \ldots a_n = k^n$ $(k > 0)$, prove that $(1+a_1)(1+a_2)\ldots(1+a_n) \geq (1+k)^n$

(ii) If $x_i > a > 0$ for $i = 1, 2, \ldots, n$ and $(x_1 - a)(x_2 - a)\ldots(x_n - a) = k^n$ $(k > 0)$, prove that $x_1x_2 \ldots x_n \geq (a+k)^n$.

10. A and G are the arithmetic mean and the geometric mean respectively of n positive real numbers $a_1, a_2, \ldots, a_n$. Prove that

(i) $(1+A)^n \geq (1+a_1)(1+a_2)\ldots(1+a_n) \geq (1+G)^n$;

(ii) if $k > 0$, $(k+A)^n \geq (k+a_1)(k+a_2)\ldots(k+a_n) \geq (k+G)^n$.

11. a, b, c are three positive real numbers in harmonic progression. Prove that $\frac{a+b}{2a-b}+\frac{c+b}{2c-b}>4$.

12. $a_1, a_2, \ldots, a_n$ are n positive real numbers in arithmetic progression. Prove that

(i) $a_1a_2\ldots a_n < (\frac{a_1+a_n}{2})^n$, (ii) $a_1^2+a_2^2+\cdots+a_n^2 > n(\frac{a_1+a_n}{2})^2$.

13. $a_1, a_2, \ldots, a_n$ are n positive real numbers in harmonic progression. Prove that

(i) $a_1+a_2+\cdots+a_n > \frac{2na_1a_n}{a_1+a_n}$, (ii) $a_1a_2\ldots a_n > (\frac{2a_1a_n}{a_1+a_n})^n$.

14. (i) If $3s = a+b+c+d$ where a, b, c, d and $s-a, s-b, s-c, s-d$ are all positive, prove that $abcd > 81(s-a)(s-b)(s-c)(s-d)$, unless $a=b=c=d$.

(ii) If $(n-1)s = a_1+a_2+\cdots+a_n$ where a_i and $s-a_i$ are all positive, prove that $a_1a_2\ldots a_n \geq (n-1)^n(s-a_1)(s-a_2)\ldots(s-a_n)$.

[**Hint.** (i) $\frac{(s-b)+(s-c)+(s-d)}{3} \geq \sqrt[3]{(s-b)(s-c)(s-d)}$

$\Rightarrow a \geq \sqrt[3]{(s-b)(s-c)(s-d)}$. Equality occurs if $b=c=d$. Consider 3 other similar inequalities.]

15. (a) If $a_1, a_2, \ldots, a_n$ be all positive real numbers, prove that

(i) $(a_1+a_2+\cdots+a_n)(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}) \geq n^2$,

(ii) $(\frac{a_1+a_2+\cdots+a_n}{n})^n \geq a_1a_2(\frac{a_3+a_4+\cdots+a_n}{n-2})^{n-2}$,

(iii) $(a_1^q+a_2^q+\cdots+a_n^q)^p > n^{p-q}(a_1^p+a_2^p+\cdots+a_n^p)^q$, where p, q are rational numbers and $0<p<q$.

(b) If $a_1, a_2, \ldots, a_n$ be all positive real numbers, $S_m = a_1^m+a_2^m+\cdots+a_n^m$, $P_m = \Sigma a_1a_2\ldots a_m$ (the sum of the products taken m at a time) prove that $(n-1)!.S_m \geq (n-m)!.m!.P_m$.

[**Hint.** (b) $a_1^m+a_2^m+\cdots+a_m^m \geq m(a_1a_2\ldots a_m)$, $a_1^m+a_3^m+\cdots+a_{m+1}^m \geq m(a_1a_3\ldots a_{m+1})$, ... Consider ${}^nC_m$ such inequalities and add. R.H.S.$=m.P_m$. Each a_i^m occurs ${}^{n-1}C_{m-1}$ times in the sum in L.H.S. So ${}^{n-1}C_{m-1}.S_m \geq m.P_m$.]

16. If $a_1, a_2, \ldots, a_n$ be all positive real numbers and $s = a_1+a_2+\cdots+a_n$, prove that

(i) $\frac{s}{s-a_1}+\frac{s}{s-a_2}+\cdots+\frac{s}{s-a_n} \geq \frac{n^2}{n-1}$,

(ii) $(\frac{s-a_1}{n-1})(\frac{s-a_2}{n-1})\ldots(\frac{s-a_n}{n-1}) > a_1a_2\ldots a_n$, unless $a_1=a_2=\cdots=a_n$.

[**Hint.** (i) Consider $(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\cdots+\frac{s}{s-a_n})(\frac{s-a_1}{s}+\frac{s-a_2}{s}+\cdots+\frac{s-a_n}{s})$ and use 15(a)(i).

17. If a and b are positive rational numbers, prove that

$a^ab^b \geq (\frac{a+b}{2})^{a+b} \geq a^bb^a$.

18. If a, b, c be positive rational numbers, prove that

(i) $(\frac{a^2+b^2+c^2}{a+b+c})^{a+b+c} \geq a^ab^bc^c \geq (\frac{a+b+c}{3})^{a+b+c}$,

(ii) $a^ab^bc^c \geq (\frac{a+b}{2})^{(a+b)/2}(\frac{b+c}{2})^{(b+c)/2}(\frac{c+a}{2})^{(c+a)/2} \geq (\frac{a+b+c}{3})^{a+b+c}$,

(iii) $(a+b)^c(b+c)^a(c+a)^b < \{\frac{2}{3}(a+b+c)\}^{a+b+c}$, unless $a=b=c$.

19. If n be a positive integer >1, prove that

(i) $n^n > (\frac{n+1}{2})^{n+1}$,

(ii) $2^{n(n+1)} > (n+1)^{n+1}(\frac{n}{1})^n(\frac{n-1}{2})^{n-1}\ldots(\frac{2}{n-1})^2.\frac{1}{n}$,

(iii) $(\frac{2n+1}{3})^{n(n+1)/2} > 1.2^2.3^3\ldots n^n > (\frac{n+1}{2})^{n(n+1)/2}$,

(iv) $\frac{1}{n+1}+\frac{1}{n+3}+\cdots+\frac{1}{3n-1} > \frac{1}{2}$.

20. If $a>0$ but $\neq 1$ and x, y, z are rational numbers in ascending order of magnitude, prove that $a^x(y-z)+a^y(z-x)+a^z(x-y) < 0$.

21. If $a_1, a_2, \ldots, a_n$ be n positive rational numbers and $s = a_1+a_2+\cdots+a_n$, prove that

(i) $a_1^{a_1}a_2^{a_2}\ldots a_n^{a_n} \geq (\frac{s}{n})^s$,

(ii) $(\frac{s}{a_1}-1)^{a_1}(\frac{s}{a_2}-1)^{a_2}\ldots(\frac{s}{a_n}-1)^{a_n} \leq (n-1)^s$,

(iii) $(\frac{s-a_1}{n-1})^{a_1}(\frac{s-a_2}{n-1})^{a_2}\ldots(\frac{s-a_n}{n-1})^{a_n} \leq (\frac{s}{n})^s$.

22. If the perimeter of a triangle remains constant, prove that the area of the triangle is greatest when the triangle is equilateral.

23. Prove that in a triangle ABC, $R \geq 2r$, where R is the circum-radius and r is the in-radius of the triangle; and $R=2r$ if the triangle is equilateral.

[**Hint.** $R = \frac{abc}{4\Delta}$ and $r = \frac{2\Delta}{a+b+c}$, where Δ is the area of the triangle.]

24. Prove that the minimum value of $x^2+y^2+z^2$ is $(\frac{c}{7})^2$ where x, y, z are positive real numbers subject to the condition $2x+3y+6z = c$, c being a constant. Find the values of x, y, z for which the minimum value is attained.

25. Find the greatest value of xyz where x, y, z are positive real numbers and

(i) $x^2+y^2+z^2 = 12$; (ii) $xy+yz+zx = 27$.

26. If p and q are positive real numbers, prove that the least value of $p^2\sec^2\theta+q^2\operatorname{cosec}^2\theta$ for different real values of θ is $(p+q)^2$.

[**Hint.** $p^2\tan^2\theta+q^2\cot^2\theta \geq 2pq$.]

27. If p and q are positive rational numbers, prove that the greatest value of $\cos^{2p}\theta\sin^{2q}\theta$ for different real values of θ is $\frac{p^pq^q}{(p+q)^{p+q}}$.

[**Hint.** Consider positive numbers $\frac{\cos^2\theta}{p}, \frac{\sin^2\theta}{q}$ with weights p, q respectively. Apply A.M. $\geq$ G.M.]

28. Prove that the minimum value of $x^2 + y^2 + z^2$ is 27, where x, y, z are positive real variables satisfying the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

29. Prove that the least value of $x+2y+4z$ is $4\sqrt{3}$, where x, y, z are positive real numbers satisfying the condition $x^2y^3z = 8$.

30. If x, y, z are positive real numbers and $x^2 + y^2 + z^2 = 12$, prove that the minimum value of $x^3 + y^3 + z^3$ is 24 and the maximum value of $x + y + z$ is 6.

31. Prove that the greatest value of a^2b^3 is $\frac{3}{16}$, where a, b are positive real numbers satisfying the condition $3a + 4b = 5$.

32. If a and b are positive constants, prove that the greatest value of $(a - x)(b - y)(bx + ay)$ is $\frac{8}{27}a^2b^2$ where x, y are real numbers satisfying the conditions $0 < x < a, 0 < y < b$. Find the values of x, y for which the greatest value is attained.

33. Find the greatest value of the following. Determine in each case the value of the variables for which the greatest value is attained.

(i) $(2x + 5)^3(5 - x)^2$ when $-\frac{5}{2} < x < 5$;

(ii) $(3x + 1)(1 - 2x)$ when $-\frac{1}{3} < x < \frac{1}{2}$;

(iii) $(x + 1)(y + 2)$ when $2x + y = 1$ and $-1 < x < \frac{3}{2}$;

(iv) $xy(1 - 2x - 3y)$ when $x > 0, y > 0, 2x + 3y < 1$;

(v) $x^2y^3(6 - x - y)$ when $x > 0, y > 0, x + y < 6$;

(vi) $xy(1 - x^2 - y^2)$ when $x > 0, x^2 + y^2 < 1, y > 0$;

(vii) xyz when $x > 0, y > 0, z > 0$ and $xy + yz + zx = 6$.

34. If a_i, b_i, c_i be all positive real numbers, use Holder's inequality to prove that

(i) $(1 + a_1{}^4)(1 + a_2{}^4)(1 + a_3{}^4)(1 + a_4{}^4) \geq (1 + a_1a_2a_3a_4)^4$;

(ii) $(a_1{}^4 + b_1{}^4)(a_2{}^4 + b_2{}^4)(a_3{}^4 + b_3{}^4)(a_4{}^4 + b_4{}^4) \geq (a_1a_2a_3a_4 + b_1b_2b_3b_4)^4$;

35. If a, b, c, p, q, r be all positive real numbers, use Holder's inequality to prove that

(i) $(a^2p^3 + b^2q^3)^5 \geq (a^5 + b^5)^2(p^5 + q^5)^3$;

(ii) $(a^2p^3 + b^2q^3 + c^2r^3)^5 \geq (a^5 + b^5 + c^5)^2(p^5 + q^5 + r^5)^3$.

36. If a, b, c, p, q, r be all positive real numbers, prove that

(i) $[(a^4+b^4+c^4)^{\frac{3}{4}} + (p^4+q^4+r^4)^{\frac{3}{4}}]^4 \geq [(a^3+p^3)^{\frac{4}{3}} + (b^3+q^3)^{\frac{4}{3}} + (c^3+r^3)^{\frac{4}{3}}]^3$;

(ii) $[(a^3+p^3)^{1/3} + (b^3+q^3)^{1/3} + (c^3+r^3)^{1/3}]^3 \geq (a+b+c)^3 + (p+q+r)^3$.

37. If a, b, c, d be all positive real numbers, prove that

$$(a^2 + b^2 + c^2 + d^2)^3 > (a^3 + b^3 + c^3 + d^3)^2.$$

2. COMPLEX NUMBERS

2.1. Introduction.

It follows from the properties of real numbers that the square of a real number is never negative. Consequently, the elementary quadratic equation $x^2 + 1 = 0$ has no solution in the system of real numbers. Introduction of a new type of numbers, called *complex numbers*, has made it possible to provide solutions of the equation $x^2 + 1 = 0$ and also of the more general type of equations

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers.

2.2. Complex numbers.

Definition. A complex number z is defined to be an *ordered pair* of real numbers (a, b) that satisfies the following condition (i) and the following laws of operations (ii) and (iii)

(i) $(a, b) = (c, d)$ if and only if $a = c, b = d$ (condition of *equality*)

(ii) $(a, b) + (c, d) = (a + c, b + d)$ (definition of addition)

(iii) $(a, b).(c, d) = (ac - bd, ad + bc)$ (definition of multiplication).

Of the ordered pair (a, b), the first component a is said to be the *real part* of z and is denoted by $Rl\ z$ and the second component b is said to be the *imaginary part* of z and is denoted by $Im\ z$.

Theorem 2.2.1. Addition of complex numbers is commutative.
This says that if z_1, z_2 be complex numbers then $z_1 + z_2 = z_2 + z_1$.

Proof. Let $z_1 = (a_1, b_1), z_2 = (a_2, b_2)$.

Then $z_1 + z_2 = (a_1 + a_2, b_1 + b_2), z_2 + z_1 = (a_2 + a_1, b_2 + b_1)$.

But $a_1 + a_2 = a_2 + a_1, b_1 + b_2 = b_2 + b_1$, since addition of real numbers is commutative.

Therefore $z_1 + z_2 = z_2 + z_1$.

Theorem 2.2.2. Addition of complex numbers is associative.
This says that if z_1, z_2, z_3 be complex numbers then

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

Proof. Let $z_1 = (a_1, b_1), z_2 = (a_2, b_2), z_3 = (a_3, b_3)$.

Then $z_1 + z_2 = (a_1 + a_2, b_1 + b_2), z_2 + z_3 = (a_2 + a_3, b_2 + b_3)$,

$(z_1 + z_2) + z_3 = [(a_1 + a_2) + a_3, (b_1 + b_2) + b_3$,

$z_1 + (z_2 + z_3) = [a_1 + (a_2 + a_3), b_1 + (b_2 + b_3)]$.

But $(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3), (b_1 + b_2) + b_3 = b_1 + (b_2 + b_3)$, since addition of real numbers is associative.

Therefore $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

Theorem 2.2.3. Multiplication of complex numbers is commutative. This says that if z_1, z_2 be complex numbers then $z_1.z_2 = z_2.z_1$.

Proof. Let $z_1 = (a_1, b_1), z_2 = (a_2, b_2)$.

Then $z_1.z_2 = (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2)$

$z_2z_1 = (a_2a_1 - b_2b_1, a_2b_1 + b_2a_1)$.

But $a_1a_2 = a_2a_1, b_1b_2 = b_2b_1, a_1b_2 = b_2a_1, b_1a_2 = a_2b_1$, since multiplication of real numbers is commutative.

Therefore $a_1a_2 - b_1b_2 = a_2a_1 - b_2b_1, a_1b_2 + b_1a_2 = a_2b_1 + b_2a_1$, since addition of real numbers is commutative.

Therefore $z_1.z_2 = z_2.z_1$.

Theorem 2.2.4. Multiplication of complex numbers is associative. This says that if z_1, z_2, z_3 be complex numbers then $(z_1.z_2).z_3 = z_1.(z_2.z_3)$.

Proof. Left to the reader.

Theorem 2.2.5. If z_1, z_2, z_3 be complex numbers then

$$z_1.(z_2 + z_3) = z_1.z_2 + z_1.z_3.$$

Proof. Let $z_1 = (a_1, b_1), z_2 = (a_2, b_2), z_3 = (a_3, b_3)$.

Then $z_1.(z_2 + z_3) = (a_1, b_1).(a_2 + a_3, b_2 + b_3)$

$= [a_1(a_2 + a_3) - b_1(b_2 + b_3), a_1(b_2 + b_3) + b_1(a_2 + a_3)]$

$= [(a_1a_2 - b_1b_2) + (a_1a_3 - b_1b_3), (a_1b_2 + b_1a_2) + (a_1b_3 + b_1a_3)]$

$= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) + (a_1a_3 - b_1b_3, a_1b_3 + b_1a_3)$

$= (a_1, b_1).(a_2, b_2) + (a_1, b_1).(a_3, b_3)$

$= z_1.z_2 + z_1.z_3$.

The complex number $(0, 0)$ satisfies the following properties–

(i) $(a, b) + (0, 0) = (a, b)$ for all complex numbers (a, b);

(ii) $(a, b).(0, 0) = (0, 0)$ for all complex numbers (a, b).

$(0, 0)$ is said to be the *zero complex number* and it is denoted by 0. Thus $z = 0$ is an abbreviation for $z = (0, 0)$.

The complex number $(1, 0)$ satisfies the property –

$(a, b).(1, 0) = (a, b)$ for all complex numbers (a, b).

$(1, 0)$ is said to be the *unit complex number* and is denoted by 1.

Negative of a complex number z_1, denoted by $-z_1$, is defined to be a complex number z_2 such that $z_1 + z_2 = 0$.

Let $z_1 = (a_1, b_1), z_2 = (a_2, b_2)$.

$z_1 + z_2 = 0 \Rightarrow (a_1 + a_2, b_1 + b_2) = (0, 0)$

$\Rightarrow a_1 + a_2 = 0, b_1 + b_2 = 0$

$\Rightarrow a_2 = -a_1, b_2 = -b_1$.

Therefore $z_2 = (-a_1, -b_1)$.

Thus $-(a_1, b_1) = (-a_1, -b_1)$.

Subtraction. Let z_1, z_2 be two complex numbers. $z_1 - z_2$ is defined to be the complex number $z_1 + (-z_2)$.

Thus $(a_1, b_1) - (a_2, b_2) = (a_1, b_1) + [-(a_2, b_2)]$

$= (a_1, b_1) + (-a_2, -b_2)$

$= (a_1 - a_2, b_1 - b_2)$.

Multiplicative inverse of a non-zero complex number z_1, denoted by z_1^{-1}, is defined to be a complex number z_2 such that $z_1.z_2 = 1$.

Multiplicative inverse of the zero complex number is not defined.

Let $z_1 = (a_1, b_1), z_2 = (a_2, b_2)$.

Then $z_1.z_2 = 1 \Rightarrow (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) = (1, 0)$

$\Rightarrow a_1a_2 - b_1b_2 = 1, a_1b_2 + a_2b_1 = 0$.

Therefore $a_2 = \frac{a_1}{a_1^2 + b_1^2}, b_2 = \frac{-b_1}{a_1^2 + b_1^2}$.

Since $(a_1, b_1) \neq (0, 0), a_2, b_2$ both exist. Therefore

$$(a_1, b_1)^{-1} = \left(\frac{a_1}{a_1^2 + b_1^2}, \frac{-b_1}{a_1^2 + b_1^2}\right).$$

Division. Let z_1, z_2 be two complex numbers where z_2 is non-zero. $\frac{z_1}{z_2}$ is defined to be the complex number $z_1.z_2^{-1}$.

Thus $\frac{(a_1, b_1)}{(a_2, b_2)} = (a_1, b_1).(a_2, b_2)^{-1}$, provided $(a_2, b_2) \neq (0, 0)$

$= (a_1, b_1).\left(\frac{a_2}{a_2^2 + b_2^2}, \frac{-b_2}{a_2^2 + b_2^2}\right)$

$= \left(\frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2}, \frac{-a_1b_2 + b_1a_2}{a_2^2 + b_2^2}\right)$.

Theorem 2.2.6. If the product of two complex numbers be zero, then at least one of them is zero.

Proof. Let $z_1 = (a_1, b_1), z_2 = (a_2, b_2)$ be two complex numbers such that $z_1 z_2 = 0$.

Then $(a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) = (0, 0)$.

Therefore $a_1a_2 - b_1b_2 = 0$ ... ... (i)

$a_1b_2 + b_1a_2 = 0$ ... ... (ii)

Let $z_2 \neq 0$. Then ${a_2}^2 + {b_2}^2 \neq 0$.

Multiplying (i) by a_2 and (ii) by b_2 and adding, we have

$$a_1({a_2}^2 + {b_2}^2) = 0.$$

Multiplying (i) by b_2 and (ii) by a_2 and subtracting, we have

$$b_1({a_2}^2 + {b_2}^2) = 0.$$

Consequently, $a_1 = b_1 = 0$. Thus $z_2 \neq 0 \Rightarrow z_1 = 0$.

By similar arguments, $z_1 \neq 0 \Rightarrow z_2 = 0$.

The complex numbers of the form $(a, 0)$ having imaginary part 0 are of special interest. For such complex numbers

$(a_1, 0) + (a_2, 0) = (a_1 + a_2, 0)$ and $(a_1, 0).(a_2, 0) = (a_1a_2, 0)$.

It is evident that the operations of addition and multiplication for such complex numbers take the form of arithmetic addition and multiplication for real numbers.

The complex number $(a, 0)$ is identified with the real number a. We write $a = (a, 0)$. In particular, $0 = (0, 0)$.

The complex numbers of the form $(0, b)$ having the real part 0 and the imaginary part non-zero are of special interest. Such a complex number is said to be an *imaginary number*. For such complex numbers

$(0, b_1) + (0, b_2) = (0, b_1 + b_2)$ and

$(0, b_1).(0, b_2) = (-b_1b_2, 0)$.

The sum of two imaginary numbers is imaginary and the product of two such numbers is real.

In particular, the complex number $(0, 1)$ is said to be the *imaginary unit* and it is denoted by i.

Thus $i.i = (0, 1).(0, 1) = (-1, 0) = -1$.

Also $(0, b) = (0, 1).(b, 0) = (b, 0).(0, 1)$.

Hence $(0, b)$ can be expressed as ib or bi.

2.3. Normal form.

The complex number (a, b) can be expressed as $(a, b) = (a, 0) + (0, b)$.

Again, $(0, b) = (0, 1).(b, 0) = (b, 0).(0, 1)$.

Therefore $(a, b) = (a, 0) + (0, 1).(b, 0) = (a, 0) + (b, 0).(0, 1)$

$= a + ib$ or $a + bi$.

Since $(a, b) + (c, d) = (a + c, b + d), (a + bi) + (c + di) = (a + c) + (b + d)i$.

Since $(a, b).(c, d) = (ac - bd, ad + bc), (a + bi).(c + di) = (ac - bd) + (ad + bc)i$.

It is observed that the sum and the product of two complex numbers $a + bi$ and $c + di$ take the same form as the sum and the product of two real binomials $a + bi$ and $c + di$ (treating i as real) if we take care always to replace i^2 by -1.

For example, $(2 + 3i).(3 + i) = (2.3 - 3.1) + (2.1 + 3.3)i$

$= 3 + 11i$, by definition.

The ordinary multiplication gives

$$\begin{aligned}(2 + 3i).(3 + i) &= 2.3 + 2.i + 3.3i + 3i^2 \\ &= 6 + 11i - 3 = 3 + 11i.\end{aligned}$$

The quotient $\frac{a+bi}{c+di}$ where $(c, d) \neq (0, 0)$ can be computed as

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(-ad+bc)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{-ad+bc}{c^2+d^2}\, i.$$

2.4. Geometrical representation.

Just as real numbers are represented as points on a line, complex numbers (which are also ordered pairs of real numbers) can be represented as points on a plane.

With respect to a given rectangular co-ordinate system in a plane, the complex number $z = a + bi$ can be represented by the point with co-ordinates (a, b).

The first co-ordinate axis is called the *real axis* and the second co-ordinate axis is called the *imaginary axis*.

The plane in which this representation is made is said to be the *complex plane*, or the *Gaussian plane*, after the name of Gauss, a celebrated German mathematician.

The origin represents the zero complex number. The points on the real axis represent all real numbers of the form $(a, 0)$ and the points on the imaginary axis, other than the origin, represent all imaginary numbers of the form $(0, b)$.

2.5. Conjugate of a complex number.

Let $z = a + bi$ be a complex number. The conjugate of z, denoted by $\bar{z}$, is defined to be the complex number $a - bi$.

Thus $\bar{z} = a - bi$ when $z = a + bi$.

Geometrically, the point $\bar{z}$ is the reflection of the point z in the real axis

Properties.

1. $\overline{\bar{z}} = z$.
2. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
3. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.
4. $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$.
5. $\left(\overline{\frac{z_1}{z_2}}\right) = \frac{\overline{z_1}}{\overline{z_2}}$ provided $z_2 \neq 0$.
6. $z + \bar{z} = 2(Rl\ z), z - \bar{z} = 2i(Im\ z)$
7. $z.\bar{z}$ is purely real.

The proofs are immediate.

2.6. Modulus of a complex number.

Let $z = a + bi$ be a complex number. The non-negative real number $\sqrt{a^2 + b^2}$ is said to be the *absolute value* or *the modulus* of z and is denoted by $| z |$, or mod z, or mod $(a + bi)$.

Geometrically, mod z is the distance of the point z from the origin in the complex plane. In general, $| z_1 - z_2 |$ is the distance between the points z_1 and z_2. It is also evident from the definition that

$$| z_1 - z_2 | = | (a_1 - a_2) + (b_1 - b_2)i | = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

Since $| z |$ is a real number, the statement $| z_1 | > | z_2 |$ has a meaning. But the statements like $z_1 > z_2, z_1 < z_2$ have no meaning unless z_1, z_2 are purely real.

It follows from definition that $| z | = | \bar{z} |$.
Also $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = | z |^2$.

Theorem 2.6.1. The modulus of the product of two complex numbers is the product of their moduli.

Proof. Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$.
Then $z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i$.

Therefore
$$\begin{aligned} |z_1 z_2| &= \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + b_1 a_2)^2} \\ &= \sqrt{{a_1}^2 {a_2}^2 + {b_1}^2 {b_2}^2 + {a_1}^2 {b_2}^2 + {b_1}^2 {a_2}^2} \\ &= \sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)} \\ &= | z_1 | \, | z_2 |. \end{aligned}$$

Alternative method.

$| z_1 z_2 |^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\bar{z_1}\bar{z_2}) = (z_1 \bar{z_1})(z_2 \bar{z_2}) = | z_1 |^2 | z_2 |^2$.

Therefore $|z_1 z_2| = |z_1||z_2|$, since $|z_1 z_2|, |z_1|, |z_2|$ are each non-negative.

Theorem 2.6.2. The modulus of the quotient of two complex numbers is the quotient of their moduli.

Proof. Left to the reader.

Theorem 2.6.3. (i) $| z | \geq (Rl\ z)$, the equality occurs when z is a non-negative real number.

(ii) $| z | \geq (Im\ z)$, the equality occurs when $z = iy$ with $y \geq 0$.

Proof. (i) Let $z = x + iy$. Then $| z | = \sqrt{x^2 + y^2}$ and $Rl\ z = x$.
Clearly, $\sqrt{x^2 + y^2} \geq x$. The equality occurs when $y = 0$ and $x \geq 0$.

(ii) Proof left to the reader.

Theorem 2.6.4. Triangle inequality.

If z_1, z_2 be two complex numbers, then $| z_1 + z_2 | \leq | z_1 | + | z_2 |$, the equality occurs if (i) one or both of z_1, z_2 be zero,

or (ii) $\frac{z_1}{z_2}$ is a positive real number.

Proof.
$$\begin{aligned} | z_1 + z_2 |^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\ &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1} \\ &= | z_1 |^2 + | z_2 |^2 + 2Rl\ (z_1\overline{z_2}), \text{ since } z_2\overline{z_1} = \overline{z_1\overline{z_2}} \\ &\leq | z_1 |^2 + | z_2 |^2 + 2 | z_1 | \, | \overline{z_2} |, \text{ since } Rlz \leq | z | \\ &\leq | z_1 |^2 + | z_2 |^2 + 2 | z_1 | \, | z_2 |, \text{ since } | \overline{z_2} | = | z_2 | \\ &= (| z_1 | + | z_2 |)^2. \end{aligned}$$

Therefore $| z_1 + z_2 | \leq | z_1 | + | z_2 |$, since $| z_1 + z_2 |, | z_1 |, | z_2 |$ are each non-negative. The equality occurs when $z_1\overline{z_2}$ is real and ≥ 0.

i.e., when either (i) one or both of z_1, z_2 be zero

or, (ii) $z_1\overline{z_2}$ is positive real which implies

$\frac{z_1}{z_2}$ is positive real, since $\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{|z_2|^2}$.

Note. This inequality can be extended to a finite number of terms.

$$| z_1 + z_2 \cdots + z_n | \leq | z_1 | + | z_2 | + \cdots + | z_n |,$$

the equality occurs if the ratio of every two non-zero complex numbers be positive real.

Corollary. $| z_1 | = | z_2 + (z_1 - z_2) | \leq | z_2 | + | z_1 - z_2 |$ and
$| z_2 | = | z_1 + (z_2 - z_1) | \leq | z_1 | + | z_1 - z_2 |$.
Therefore $| z_1 | - | z_2 | \leq | z_1 - z_2 |$ and $| z_2 | - | z_1 | \leq | z_1 - z_2 |$.
Combining, we have $|| z_1 | - | z_2 || \leq | z_1 - z_2 |$.

Worked Examples.

1. If z_1, z_2 be two complex numbers prove that

$$|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2).$$

$$\begin{aligned}|z_1+z_2|^2=(z_1+z_2)(\overline{z_1+z_2}) &= (z_1+z_2)(\overline{z_1}+\overline{z_2})\\ &= z_1\overline{z_1}+z_1\overline{z_2}+z_2\overline{z_1}+z_2\overline{z_2}\\ &= |z_1|^2+z_1\overline{z_2}+z_2\overline{z_1}+|z_2|^2\end{aligned}$$

$$\begin{aligned}|z_1-z_2|^2=(z_1-z_2)(\overline{z_1-z_2}) &= (z_1-z_2)(\overline{z_1}-\overline{z_2})\\ &= z_1\overline{z_1}-z_1\overline{z_2}-z_2\overline{z_1}+z_2\overline{z_2}\\ &= |z_1|^2-z_1\overline{z_2}-z_2\overline{z_1}+|z_2|^2\end{aligned}$$

Therefore $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$.

2. If a and b are complex numbers show that

$$|a+\sqrt{a^2-b^2}|+|a-\sqrt{a^2-b^2}|=|a+b|+|a-b|.$$

Let $z_1=a+\sqrt{a^2-b^2}$, $z_2=a-\sqrt{a^2-b^2}$.

$$\begin{aligned}(|z_1|+|z_2|)^2 &= |a+\sqrt{a^2-b^2}|^2+|a-\sqrt{a^2-b^2}|^2\\ &\quad +2|a+\sqrt{a^2-b^2}||a-\sqrt{a^2-b^2}|\\ &= |a+\sqrt{a^2-b^2}|^2+|a-\sqrt{a^2-b^2}|^2+2|b^2|,\\ &= 2|a|^2+2|\sqrt{a^2-b^2}|^2+2|b^2|, \text{ since } |u+v|^2+|u-v|^2\\ &\qquad = 2|u^2|+2|v^2|\\ &= 2(|a|^2+|b|^2)+2|a^2-b^2|, \text{ since } |u|^2=|u^2|\\ &= |a+b|^2+|a-b|^2+2|(a+b)(a-b)|\\ &= |a+b|^2+|a-b|^2+2|(a+b)||(a-b)|\\ &= (|a+b|+|a-b|)^2.\end{aligned}$$

Therefore $|z_1|+|z_2|=|a+b|+|a-b|$, since both sides are non-negative.

3. z is a complex number satisfying the condition $|z-\frac{3}{z}|=2$. Find the greatest and the least value of $|z|$.

We have $||z|-|\frac{3}{z}||\le|z-\frac{3}{z}|$. Therefore $||z|-|\frac{3}{z}||\le 2$

or, $|r-\frac{3}{r}|\le 2$, where $|z|=r$. Clearly, $r>0$ in this case.

This implies $-2r\le r^2-3\le 2r$, since $r>0$.

$r^2-2r-3\le 0\Rightarrow(r-3)(r+1)\le 0\Rightarrow 0<r\le 3$, since $r>0$.

$r^2+2r-3\ge 0\Rightarrow(r+3)(r-1)\ge 0\Rightarrow r\ge 1$, since $r>0$.

Therefore $1\le r\le 3$. The greatest value of $|z|$ is 3 and the least value of $|z|$ is 1.

2.7. Polar form.

Let $z=a+bi$ be a complex number. In the complex plane z is represented by the point whose Cartesian co-ordinates are (a,b) referred to two perpendicular lines as axes, the first co-ordinate axis being called the real axis and the second the imaginary axis.

Taking the origin as the pole and the real axis as the initial line, let (r,θ) be the polar co-ordinates of the point (a,b). Then $a=r\cos\theta, b=r\sin\theta$.

Geometrically, r is the distance of the point (a,b) from the origin. r is said to be the *modulus* of the complex number z.

For every point other than the origin, $r>0$. For the point $(0,0)$, $r=0$. Thus every non-zero complex number has a positive modulus.

θ is the angle made by the radius vector through the point (a,b) with the real axis. θ is called *an argument* or *an amplitude* of z.

θ cannot be determined for the zero complex number.

Consequently, a non-zero complex number z is represented in the form $z=r(\cos\theta+i\sin\theta)$.

This is called the *polar form* or *modulus-amplitude form* of the complex number z.

As θ is indeterminate for the zero complex number, the zero complex number has no polar representation.

For a non-zero complex number z, θ has infinitely many values differing from one another by a multiple of 2π.

If $z=r(\cos\theta+i\sin\theta)$, r is determined from the relations $r\cos\theta=a, r\sin\theta=b$ giving $r=\sqrt{a^2+b^2}$ and θ is determined from the relations $\cos\theta=\frac{a}{r}$, $\sin\theta=\frac{b}{r}$.

All values of θ are expressed as Arg z (or Amp z). If α be a value of θ satisfying the relations $\cos\theta=\frac{a}{r}$, $\sin\theta=\frac{b}{r}$ then Arg z (Amp z) $=\alpha+2n\pi$, n being an integer.

The *principal* argument (amplitude) of z, denoted by arg z (amp z), is defined to be the angle θ which satisfies the inequality $-\pi<\theta\le\pi$.

For example, $\frac{3\pi}{2}$ is an argument of the complex number $-i$ but it is not the principal argument because $\theta(=\frac{3\pi}{2})$ does not satisfy the relation $-\pi<\theta\le\pi$. The principal argument of $-i$ is $-\frac{\pi}{2}$ and Arg $(-i)=\frac{3\pi}{2}+2n\pi$, or $-\frac{\pi}{2}+2m\pi$, where n and m are integers.

Note. An argument of the complex number $a+bi$ is to be determined from the relations $\cos\theta=\frac{a}{r}$, $\sin\theta=\frac{b}{r}$ simultaneously and not from the single relation $\theta=\tan^{-1}\frac{b}{a}$.

Worked Examples.

1. Find mod z and arg z and express z in polar form, where
(i) $z = -1+i$, (ii) $z = 1-i$.

(i) Let $-1+i = r(\cos\theta + i\sin\theta)$. Then $r\cos\theta = -1, r\sin\theta = 1$.
We have $r^2 = 2$ and therefore $r = \sqrt{2}$.
Therefore $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$ and these determine $\theta = \frac{3\pi}{4}$.
Therefore mod $z = \sqrt{2}$ and arg $z = \frac{3\pi}{4}$.
Hence $-1+i = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$.

(ii) Let $1-i = r(\cos\theta + i\sin\theta)$. Then $r\cos\theta = 1, r\sin\theta = -1$.
We have $r^2 = 2$ and therefore $r = \sqrt{2}$.
Therefore $\cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = -\frac{1}{\sqrt{2}}$ and these determine $\theta = -\frac{\pi}{4}$.
Therefore mod $z = \sqrt{2}$ and arg $z = -\frac{\pi}{4}$.
Hence $1-i = \sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$.

2. Express $-1-i$ in polar form.

Let $-1-i = r(\cos\theta + i\sin\theta)$. Then $r\cos\theta = -1, r\sin\theta = -1$.
We have $r^2 = 2$ and therefore $r = \sqrt{2}$.
Therefore $\cos\theta = -\frac{1}{\sqrt{2}}, \sin\theta = -\frac{1}{\sqrt{2}}$. These determine $\theta = \frac{5\pi}{4}$.
Hence $-1-i = \sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$.

Note. Here $\theta = \frac{5\pi}{4}$ does not give the principal argument of $-1-i$. The principal argument is $-\frac{3\pi}{4}$. Using the principal argument, the polar form of $-1-i$ is $\sqrt{2}[\cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})]$.

3. Let us recall the definition of $\tan^{-1}x$, when x is a real number.

$\tan^{-1}x$ is the *unique* angle θ satisfying the relations
(i) $\tan\theta = x$ and (ii) $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

We shall now show by examples that an argument of the complex number $z = a+bi = r(\cos\theta + i\sin\theta)$ cannot be determined from the relation $\theta = \tan^{-1}\frac{b}{a}$, in general.

(i) Let $a = 1, b = 1$. Then $r = \sqrt{2}$ and $\cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$.
The arguments are $\frac{\pi}{4} + 2k\pi$, where k is an integer.
Here $\tan^{-1}\frac{b}{a} = \tan^{-1}1 = \frac{\pi}{4}$.

Hence $\tan^{-1}\frac{b}{a}$ gives an argument of z.

(ii) Let $a = -1, b = 1$. Then $r = \sqrt{2}$ and $\cos\theta = -\frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$.
The arguments are $\frac{3\pi}{4} + 2k\pi$, where k is an integer.
Here $\tan^{-1}\frac{b}{a} = \tan^{-1}(-1) = -\frac{\pi}{4}$, but $-\frac{\pi}{4}$ is not an argument of z.

Hence $\tan^{-1}\frac{b}{a}$ does not give an argument of z.

(iii) Let $a = -2, b = 0$. Then $r = 2$ and $\cos\theta = -1$, $\sin\theta = 0$.
The arguments are $\pi + 2k\pi$, where k is an integer.
Here $\tan^{-1}\frac{b}{a} = \tan^{-1}0 = 0$, but 0 is not an argument of z.

Hence $\tan^{-1}\frac{b}{a}$ does not give an argument of z.

(iv) Let $a = -1, b = -1$. Then $r = \sqrt{2}$, $\cos\theta = -\frac{1}{\sqrt{2}}$, $\sin\theta = -\frac{1}{\sqrt{2}}$.
The arguments are $\frac{-3\pi}{4} + 2k\pi$, where k is an integer.
Here $\tan^{-1}\frac{b}{a} = \tan^{-1}1 = \frac{\pi}{4}$, but $\frac{\pi}{4}$ is not an argument of z.

Hence $\tan^{-1}\frac{b}{a}$ does not give an argument of z.

4. Some important complex numbers and their polar forms with principal argument.

(i) $1 = \cos 0 + i\sin 0$, mod $1 = 1$, arg $1 = 0$;
(ii) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, mod $i = 1$, arg $i = \frac{\pi}{2}$;
(iii) $-1 = \cos\pi + i\sin\pi$, mod $(-1) = 1$, arg $(-1) = \pi$;
(iv) $-i = \cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})$, mod $(-i) = 1$, arg $(-i) = -\frac{\pi}{2}$.

Theorem 2.7.1. Let z_1, z_2 be two non-zero complex numbers. If θ_1 be an argument of z_1 and θ_2 be an argument of z_2 then $\theta_1 + \theta_2$ is an argument of z_1z_2.

Proof. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$
$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin\theta_1 + \theta_2)]$.
This shows that $\theta_1 + \theta_2$ is an argument of z_1z_2. □

Note. The theorem does not hold, in general, if the principal arguments be considered. That is, $\arg(z_1z_2) \neq \arg z_1 + \arg z_2$, in general.

For example, let $z_1 = 1-i, z_2 = i, z_3 = -1+i$.
Then arg $z_1 = -\frac{\pi}{4}$, arg $z_2 = \frac{\pi}{2}$, arg $z_3 = \frac{3\pi}{4}$.
$z_1z_2 = 1+i, z_2z_3 = -1-i$, arg $(z_1z_2) = \frac{\pi}{4}$, arg $(z_2z_3) = -\frac{3\pi}{4}$.

Here arg $(z_1z_2) = \arg z_1 + \arg z_2$, but arg $(z_2z_3) \neq \arg z_2 + \arg z_3$.

Theorem 2.7.2. Let z_1, z_2 be two non-zero complex numbers. If θ_1 be an argument of z_1 and θ_2 be an argument of z_2 then $\theta_1 - \theta_2$ is an argument of $\frac{z_1}{z_2}$.

Proof. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.
$\frac{z_1}{z_2} = \frac{r_1\cos\theta_1 + i\sin\theta_1}{r_2\cos\theta_2 + i\sin\theta_2}$

$= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)$

$= \frac{r_1}{r_2}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)$

$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$.

This shows that $\theta_1 - \theta_2$ is an argument of $\frac{z_1}{z_2}$.

Note. The theorem does not hold, in general, if the principal arguments be considered. That is, $\arg\left(\frac{z_1}{z_2}\right) \neq \arg z_1 - \arg z_2$, in general.

For example, let $z_1 = 1 - i, z_2 = i, z_3 = -1 + i$.

Then $\arg z_1 = -\frac{\pi}{4}$, $\arg z_2 = \frac{\pi}{2}$, $\arg z_3 = \frac{3\pi}{4}$.

$\frac{z_1}{z_2} = -1 - i$, $\frac{z_1}{z_3} = -1$, $\arg\frac{z_1}{z_2} = -\frac{3\pi}{4}$, $\arg\left(\frac{z_1}{z_3}\right) = \pi$.

Here $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$, but $\arg\left(\frac{z_1}{z_3}\right) \neq \arg z_1 - \arg z_3$.

Theorem 2.7.3. If z_1, z_2 be non-zero complex numbers, then $\arg(z_1z_2) = \arg z_1 + \arg z_2 + 2k\pi$, where

$k = 0$ if $-\pi < \arg z_1 + \arg z_2 \leq \pi$,
$k = 1$ if $\arg z_1 + \arg z_2 \leq -\pi$,
$k = -1$ if $\arg z_1 + \arg z_2 > \pi$.

Proof. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where $-\pi < \theta_1 \leq \pi, -\pi < \theta_2 \leq \pi$. Then $\arg z_1 = \theta_1$, $\arg z_2 = \theta_2$.

$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

$\theta_1 + \theta_2$ is argument of z_1z_2. But $\theta_1 + \theta_2$ may not be the principal argument.

$\arg(z_1z_2) = \theta_1 + \theta_2 + 2k\pi$ where k is an integer such that $-\pi < \theta_1 + \theta_2 + 2k\pi \leq \pi$. But $-2\pi < \theta_1 + \theta_2 \leq 2\pi$.

Three cases arise.

(i) If $-\pi < \theta_1 + \theta_2 \leq \pi$, then $k = 0$.

(ii) If $\pi < \theta_1 + \theta_2 \leq 2\pi$, then $-\pi < \theta_1 + \theta_2 - 2\pi \leq 0$.
Therefore $k = -1$.

(iii) If $-2\pi < \theta_1 + \theta_2 \leq -\pi$, then $0 < \theta_1 + \theta_2 + 2\pi \leq \pi$.
Therefore $k = 1$.

Theorem 2.7.4. If z_1, z_2 be non-zero complex numbers, then $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2k\pi$, where

$k = 0$ if $-\pi < \arg z_1 - \arg z_2 \leq \pi$,
$k = 1$ if $\arg z_1 - \arg z_2 \leq -\pi$,
$k = -1$ if $\arg z_1 - \arg z_2 > \pi$.

Proof. Left to the reader.

Worked Examples (continued).

5. Find $\arg z$ where $z = 1 + i\tan\frac{3\pi}{5}$.

Let $1 + i\tan\frac{3\pi}{5} = r(\cos\theta + i\sin\theta)$. Then $r\cos\theta = 1, r\sin\theta = \tan\frac{3\pi}{5}$.

We have $r^2 = \sec^2\frac{3\pi}{5}$ and this gives $r = -\sec\frac{3\pi}{5}$, since $\sec\frac{3\pi}{5} < 0$.

Therefore $\cos\theta = -\cos\frac{3\pi}{5}$, $\sin\theta = -\sin\frac{3\pi}{5}$.

These determine $\theta = \pi + \frac{3\pi}{5}$. Since $\theta > \pi$, θ does not give the principal argument.

Therefore $\arg z = \theta - 2\pi = -\frac{2\pi}{5}$.

6. Find $\arg z$ where $z = 1 + \cos 2\theta + i\sin 2\theta$, $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

Let $z = r(\cos\phi + i\sin\phi)$. Then $r\cos\phi = 1 + \cos 2\theta, r\sin\phi = \sin 2\theta$

We have $r^2 = (1 + \cos 2\theta)^2 + \sin^2 2\theta = 4\cos^2\theta$.

This gives $r = -2\cos\theta$ since $\cos\theta < 0$.

Therefore $\cos\phi = \frac{1 + \cos 2\theta}{-2\cos\theta} = -\cos\theta$ and

$\sin\phi = \frac{\sin 2\theta}{-2\cos\theta} = -\sin\theta = \sin(\pi + \theta)$.

These determine $\phi = \pi + \theta$. $\frac{\pi}{2} < \theta < \frac{3\pi}{2} \Rightarrow \frac{3\pi}{2} < \phi < \frac{5\pi}{2}$.

As $\phi > \frac{3\pi}{2}$, $\arg z \neq \phi$.

Therefore $\arg z = \phi + 2k\pi$, where k is an integer such that $-\pi < \phi + 2k\pi \leq \pi$.

Clearly, $k = -1$ and $\arg z = \phi - 2\pi = \theta - \pi$.

7. Prove that $\arg z - \arg(-z) = \pm\pi$, according as $\arg z > 0$ or < 0.

Let $z = r(\cos\theta + i\sin\theta)$, where $-\pi < 0 \leq \pi$.

Then $\arg z = \theta$ and $-z = -r(\cos\theta + i\sin\theta)$

$= r[\cos(\pi + \theta) + i\sin(\pi + \theta)]$.

$\pi + \theta$ is an argument of $(-z)$. Let $\phi = \pi + \theta$.

Case I. Let $\arg z > 0$. Then $0 < \theta \leq \pi$ and therefore $\pi < \pi + \theta \leq 2\pi$, i.e., $\pi < \phi \leq 2\pi$.

As $\phi > \pi$, ϕ is not the principal argument of $(-z)$.

$\arg(-z) = \phi + 2k\pi$ where k is an integer such that $-\pi < \phi + 2k\pi \leq \pi$.

Clearly, $k = -1$ and $\arg(-z) = \phi - 2\pi = \theta - \pi$.

Hence $\arg z - \arg(-z) = \pi$.

Case II. Let $\arg z < 0$. Then $-\pi < \theta < 0$ and therefore $0 < \phi < \pi$.

Therefore $\arg(-z) = \phi = \pi + \theta$.

Hence $\arg z - \arg(-z) = -\pi$.

2.8. Integral and rational powers.

Let z be a complex number and n be an integer.

We define (i) $z^0 = 1$,
(ii) $z^n = z^{n-1}.z$ if $n > 0$,
(iii) $z^{-n} = (z^{-1})^n$ if $z \neq 0$ and $n > 0$.

The laws of indices for complex numbers
(i) $z^m.z^n = z^{m+n}$,
(ii) $(z^m)^n = z^{mn}$,
(iii) $z_1{}^m.z_2{}^m = (z_1 z_2)^m$
hold when m, n are integers, with proper restrictions on z, z_1, z_2 in case of negative index.

Let z be a non-zero complex number and q be a positive integer > 1. Then there is a non-zero complex number w such that $w^q = z$. The existence of such a w and its non-uniqueness will be discussed later.

Definition. Let z be a non-zero complex number and q be a positive integer. Then $z^{1/q}$ is a complex number w such that $w^q = z$. w is said to be a qth root of z.

Definition. Let z be a non-zero complex number and r be a positive rational number, say $r = \frac{p}{q}$, where p, q are positive integers, then
(i) $z^r = z^{p/q} = (z^{1/q})^p$, (ii) $z^{-r} = (\frac{1}{z})^r$.

Theorem 2.8.1. De Moivre's theorem

When n is an integer, positive or negative, and θ is a real number
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$;
when n is a fraction, positive or negative, and θ is a real number
$\cos n\theta + i \sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Proof. **Case I.** Let n be a positive integer.

The theorem holds for $n = 1$, since $(\cos + i\sin\theta)^1 = \cos\theta + i\sin\theta = \cos 1\theta + i\sin 1\theta$.

Let us assume that the theorem holds for $n = m$, where m is a positive integer.

Then $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$.

Therefore $(\cos\theta + i\sin\theta)^{m+1} = (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$
$= (\cos m\theta\cos\theta - \sin m\theta\sin\theta) + i(\cos m\theta\sin\theta + \sin m\theta\cos\theta)$
$= \cos(m+1)\theta + i\sin(m+1)\theta$.

This shows that the theorem holds for $n = m + 1$ if we assume it to hold for $n = m$. And the theorem holds for $n = 1$.

By the principle of induction, the theorem holds for all positve integers n.

Case 2. Let n be a negative integer.

Let $n = -m$, where m is positive integer.

Now $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$
$= \frac{1}{(\cos\theta + i\sin\theta)^m}$.
$= \frac{1}{\cos m\theta + i\sin m\theta}$, by case 1
$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$
$= \cos m\theta - i\sin m\theta$
$= \cos(-m)\theta + i\sin(-m)\theta$
$= \cos n\theta + i\sin n\theta$.

Case 3. Let n be a fraction, positive or negative.

Let $n = p/q$ where p, q are integers and $q > 1, p$ may be positive or negative.

Since $(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q})^q = \cos p\theta + i\sin p\theta$, by case 1
$= (\cos\theta + i\sin\theta)^p$, by case 1 or case 2,

it follows that $\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$, i.e., $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Corollary. When n is a positive or a negative integer and θ is a real number
$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$;
when n is a fraction and θ is a real number
$(\cos n\theta - i\sin n\theta)$ is one of the values of $(\cos\theta - i\sin\theta)^n$.
This follows from the relation
$(\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$.

Note. The generalised form of De Moivre's theorem states that if n be a real number and θ is real, $(\cos n\theta + i\sin n\theta)$ is a value of $(\cos\theta + i\sin\theta)^n$.

2.9. Roots of a complex number.

When n is a fraction, positive or negative, De Moivre's theorem states that $(\cos n\theta + i\sin n\theta)$ is one of the values of $(\cos\theta + i\sin\theta)^n$. It is natural to ask how many values of $(\cos\theta + i\sin\theta)^n$ do exist in that case.

We have the following theorem in this respect.

Theorem 2.9.1. If z be a non-zero complex number and n be a positive integer then there are n distinct values of $z^{1/n}$.

Proof. Let $z = r(\cos\theta + i\sin\theta)$, where $r > 0, -\pi < 0 \leq \pi$.

Let $z^{1/n} = w = \rho(\cos\phi + i\sin\phi)$.

This implies $w^n = z$ and therefore $\rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$.

This gives $\rho^n\cos n\phi = r\cos\theta$ and $\rho^n\sin n\phi = r\sin\theta$.

We have $\rho^n = r$, i.e., $\rho = \sqrt[n]{r}$ where $\sqrt[n]{r}$ is the positive nth root of r; and $n\phi = \theta + 2k\pi$, where k is an integer. Therefore $\phi = \frac{2k\pi+\theta}{n}$.

Hence $w = \sqrt[n]{r}(\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n})$, where k is an integer.

For different values of k, we get different values of w and this suggests that there are infinitely many values of $z^{1/n}$.

We now prove that these are not all distinct.

Let z_m denote the value of w corresponding to $k = m$, i.e., $z_m = \sqrt[n]{r}(\cos\frac{2m\pi+\theta}{n} + i\sin\frac{2m\pi+\theta}{n})$.

We prove that only n values $z_0, z_1, \ldots, z_{n-1}$ are distinct and every other value of w (as k runs through all other integers) takes one of these distinct values.

To prove that $z_0, z_1, \ldots, z_{n-1}$ are distinct, let us assume the contrary that $z_p = z_q$ where $0 \leq p \leq n-1, 0 \leq q \leq n-1$ and $p > q$.

$z_p = z_q$ implies $\frac{2p\pi+\theta}{n} - \frac{2q\pi+\theta}{n} = 2s\pi$, where s is an integer.

That is, $\frac{p-q}{n}$ is an integer, which is a possibility only when $p = q$, since $p - q < n$. This proves our assertion.

Let z_t be any value of w where t is other than $0, 1, \ldots, n-1$.

By division algorithm there exist integers λ and μ such that $t = n\lambda + \mu$, where $0 \leq \mu \leq n-1$.

Therefore $z_t = \sqrt[n]{r}(\cos\frac{2t\pi+\theta}{n} + i\sin\frac{2t\pi+\theta}{n})$

$= \sqrt[n]{r}[\cos(2\lambda\pi + \frac{2\mu\pi+\theta}{n}) + i\sin(2\lambda\pi + \frac{2\mu\pi+\theta}{n})]$

$= \sqrt[n]{r}(\cos\frac{2\mu\pi+\theta}{n} + i\sin\frac{2\mu\pi+\theta}{n}) = z_\mu$.

Since $0 \leq \mu \leq n-1$, it follows that there are only n distinct values of $z^{1/n}$ and they are

$\sqrt[n]{r}(\cos\frac{2\kappa\pi+\theta}{n} + i\sin\frac{2\kappa\pi+\theta}{n})$, where $k = 0, 1, \ldots, n-1$.

Note 1. They have the same modulus and their arguments are

$\frac{\theta}{n}, \frac{2\pi}{n} + \frac{\theta}{n}, \frac{4\pi}{n} + \frac{\theta}{n}, \cdots, (2\pi - \frac{2\pi}{n}) + \frac{\theta}{n}$.

Note 2. The value corresponding to $k = 0$ is called the *principal* nth root of z. The principal nth root of z is $\sqrt[n]{r}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$, where $r = \text{mod } z, \theta = \arg z$ (principal argument).

Theorem 2.9.2. If z is a non-zero complex number and m, n are positive integers prime to each other, then $(z^{1/n})^m = (z^m)^{1/n}$.

Proof. Let $z = r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leq \pi$. Then

$(z^{1/n})^m = (\sqrt[n]{r})^m[\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}]^m$, where $k = 0, 1, \ldots, n-1$

$= (\sqrt[n]{r})^m[(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})\omega^k]^m$, where $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

$= \sqrt[n]{r^m}[\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n}]\omega^{km}$, where $k = 0, 1, \ldots, n-1$.

$(z^m)^{1/n} = \sqrt[n]{r^m}[\cos\frac{2k\pi+m\theta}{n} + i\sin\frac{2k\pi+m\theta}{n}]$

$= (\sqrt[n]{r^m}[\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n}]\omega^k$, where $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and $k = 0, 1, \ldots, n-1$.

Since m and n are prime to each other, the numbers $0, m, 2m, \ldots, (n-1)m$ when divided by n, leave the remainders $0, 1, 2, \ldots, n-1$ in some order. Therefore as k runs through the values $0, 1, 2, \ldots, n-1$, the two sets of values of ω^{km} and ω^k are same.

Therefore $(z^{1/n})^m = (z^m)^{1/n}$.

Note 1. The n numbers in either set are the values of $z^{m/n}$.

Therefore $z^{m/n} = \sqrt[n]{r^m}[\cos\frac{(2k\pi+\theta)m}{n} + i\sin\frac{(2k\pi+\theta)m}{n}]$

$= \sqrt[n]{r^m}(\cos[(2k\pi+\theta)\frac{m}{n}] + i\sin[(2k\pi+\theta)\frac{m}{n}])$, where $k = 0, 1, \ldots, n-1$.

Note 2. If z is a non-zero complex number and m, n are positive integers prime to each other, then $z^{-\frac{m}{n}} = (\frac{1}{z})^{\frac{m}{n}} = [(\frac{1}{z})^m]^{\frac{1}{n}} = (z^{-m})^{\frac{1}{n}}$.

Also $z^{-\frac{m}{n}} = (\frac{1}{z})^{\frac{m}{n}} = [(\frac{1}{z})^{\frac{1}{n}}]^m = (z^{-\frac{1}{n}})^m$.

Note 3. If z is a non-zero complex number and m, n are positive integers, then $(z^{1/n})^m \neq (z^m)^{1/n}$, in general.

For example, let $z = i, m = 4, n = 6$. Then

$(z^m)^{1/n} = 1^{\frac{1}{6}} = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}$, where $k = 0, 1, \ldots, 5$. These have six distinct values.

$(z^{1/n})^m = (i^{\frac{1}{6}})^4 = \cos\frac{4r\pi+\pi}{3} + i\sin\frac{4r\pi+\pi}{3}$, where $r = 0, 1, \ldots, 5$. These have three distinct values.

Theorem 2.9.3. If z is a non-zero complex number and p, q, m, n are positive integers where $\frac{p}{q} = \frac{m}{n}$ with $gcd(m, n) = 1$, then $z^{p/q} = z^{m/n}$.

Proof. Here $np = qm$. Let $w = z^{p/q}$. Then $w = (z^{1/q})^p$, by definition.

$w^n = \{(z^{1/q})^p\}^n = (z^{1/q})^{pn} = (z^{1/q})^{qm} = \{(z^{1/q})^q\}^m = z^m$.

Therefore $w = (z^m)^{1/n} = (z^{1/n})^m$, since $gcd(m,n) = 1$
$= z^{m/n}$.

Therefore $z^{p/q} = z^{m/n}$.

This completes the proof.

Note. The theorem gives a method for evaluating $z^{p/q}$, where p, q are positive integers not prime to each other.

$z^{p/q} = (z^{\frac{1}{n}})^m = (z^m)^{1/n}$, where $\frac{p}{q} = \frac{m}{n}$ and m, n are positive integers prime to each other.

For example, $i^{\frac{4}{3}} = (i^{\frac{1}{3}})^4$, by definition. [art. 2.8]

And also $i^{\frac{4}{3}} = i^{\frac{2}{3}} = (i^{\frac{1}{3}})^2 = (i^2)^{\frac{1}{3}}$, by the theorem.

It follows that $i^{\frac{4}{3}}$ has three distinct values.

It is worthwhile to note that $(i^4)^{\frac{1}{3}}$ has six distinct values.

2.10. nth roots of unity.

When the complex number is 1, the roots are of special interest. They are called the nth *roots of unity.*

When $z = 1$,mod $z = 1$ and arg $z = 0$. Therefore by the previous theorem, the n nth roots of unity are

$\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$, where $k = 0, 1, \ldots, n-1$.

They have the same modulus 1 and their arguments are $0, \frac{2\pi}{n}, \frac{4\pi}{n}, \cdots, (2\pi - \frac{2\pi}{n})$.

Geometrically they represent points $P_0, P_1, P_2, \ldots, P_{n-1}$ in the complex plane such that $OP_0 = OP_1 = \cdots = OP_{n-1} = 1$ and $\angle P_0OP_1 = \angle P_1OP_2 = \cdots = \angle P_{n-1}OP_0 = \frac{2\pi}{n}$. Therefore $P_0, P_1, P_2, \ldots, P_{n-1}$ are the vertices of a regular polygon.

$\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ is purely real if and only if $\frac{2k}{n}$ is an integer. Since $0 \le k \le n-1$, $\frac{2k}{n}$ is an integer if and only if $k = 0, \frac{n}{2}$.

If n be odd, there is only one real nth root of unity, corresponding to $k = 0$ and the real root is 1.

If n be even, there are two real roots corresponding to $k = 0, \frac{n}{2}$. The real roots are 1 and -1.

Again the roots corresponding to $k = r$ and $k = n - r$ are $\cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}$ and $\cos \frac{2r\pi}{n} - i \sin \frac{2r\pi}{n}$ respectively. They are conjugate as well as reciprocal to each other.

When n is odd, the roots can be exhibited as $1, \cos \frac{2k\pi}{n} \pm i \sin \frac{2k\pi}{n}$, where $k = 1, 2, \ldots, \frac{n-1}{2}$.

When n is even, the roots can be exhibited as $\pm 1, \cos \frac{2k\pi}{n} \pm i \sin \frac{2k\pi}{n}$, where $k = 1, 2, \ldots, \frac{n}{2} - 1$.

When n is odd, $x^n - 1$ can be expressed as the product

$(x-1) \prod_{k=1}^{(n-1)/2} [(x - \cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n})(x - \cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n})]$

$= (x-1) \prod_{k=1}^{(n-1)/2} (x^2 - 2x \cos \frac{2k\pi}{n} + 1)$.

When n is even, $x^n - 1$ can be expressed as the product

$(x^2-1) \prod_{k=1}^{(n-2)/2} [(x - \cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n})(x - \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n})]$

$= (x^2-1) \prod_{k=1}^{(n-2)/2} [(x^2 - 2x \cos \frac{2k\pi}{n} + 1)$.

Let us denote $\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ by ω. Then the nth roots of unity are ω^k, where $k = 0, 1, \ldots, n-1$. So the roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

Let z be an arbitrary non-zero complex number $r(\cos \theta + i \sin \theta)$, where $-\pi < \theta \le \pi$.

The n nth roots of z are

$\sqrt[n]{r}(\cos \frac{2k\pi+\theta}{n} + i \sin \frac{2k\pi+\theta}{n})$, where $k = 0, 1, \ldots, n-1$.

$= \sqrt[n]{r}(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n})(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n})$

$= \sqrt[n]{r}(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n})\omega^k$

$= z_0\omega^k$ where $z_0 = \sqrt[n]{r}(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n})$, the principal nth root of z.

Thus all the nth roots of z can be expressed as $z_0, z_0\omega, z_0\omega^2, \ldots, z_0\omega^{n-1}$ and they can be obtained by multiplying the n nth roots of unity by the principal nth root of z.

Note. If $\sqrt{z}$ denote the principal square root of z then the square roots of z are $\pm\sqrt{z}$.

If $\sqrt[3]{z}$ denote the principal cube root of z then the cube roots of z are $\sqrt[3]{z}, \omega\sqrt[3]{z}$ and $\omega^2\sqrt[3]{z}$, where $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$.

Worked Examples.

1. Find the cube roots of 1.

Let $x = 1^{1/3}$. The polar form of 1 is $\cos 0 + i \sin 0$.

Then $x = (\cos 0 + i \sin 0)^{1/3}$.

There are three distinct values of x and they are $\cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}$, where $k = 0, 1, 2$; i.e., $(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})^k$, where $k = 0, 1, 2$.

Let $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Then $x = 1, \omega, \omega^2$.

Hence the cube roots of 1 are $1, \omega, \omega^2$, where $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$.

2. Find the cube roots of -1.

The polar form of -1 is $\cos \pi + i \sin \pi$.

There are three distinct values of $(-1)^{1/3}$ and they are $\cos \frac{2k\pi+\pi}{3} + i \sin \frac{2k\pi+\pi}{3}$, where $k = 0, 1, 2$.
i.e., $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$, $\cos \pi + i \sin \pi$, $\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$.

Note. The principal cube root of -1 is $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1+\sqrt{3}i}{2}$.

3. Find the fourth roots of 1.

Let $x = 1^{1/4}$.

$$\begin{aligned} \text{Then } x &= (\cos 0 + i \sin 0)^{1/4} \\ &= (\cos \tfrac{2k\pi}{4} + i \sin \tfrac{2k\pi}{4}), \text{ where } k = 0, 1, 2, 3 \\ &= (\cos \tfrac{2\pi}{4} + i \sin \tfrac{2\pi}{4})^k, \text{ where } k = 0, 1, 2, 3 \\ &= 1, i, i^2, i^3, \text{ since } \cos \tfrac{2\pi}{4} + i \sin \tfrac{2\pi}{4} = i. \end{aligned}$$

4. Solve the equation $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

$(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)(x - 1) = x^7 - 1$.

The roots of $x^7 - 1 = 0$ are $\cos \frac{2k\pi}{7} + i \sin \frac{2k\pi}{7}$, where $k = 0, \ldots, 6$.
i.e., $1, \alpha, \alpha^2, \ldots, \alpha^6$, where $\alpha = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$.

Excluding 1 (since 1 is the root of $x - 1 = 0$), the roots of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ are $\alpha, \alpha^2, \ldots, \alpha^6$, where $\alpha = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$.

5. Find the sum of 99th powers of the roots of the equation $x^7 - 1 = 0$.

The roots of $x^7 - 1 = 0$ are $\cos \frac{2r\pi}{7} + i \sin \frac{2r\pi}{7}$, $r = 0, 1, 2, \ldots, 6$.
Let $\alpha = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$. Then the roots are $1, \alpha, \alpha^2, \ldots, \alpha^6$.
$1 + \alpha^{99} + \cdots + (\alpha^6)^{99} = \frac{1-\alpha^{99.7}}{1-\alpha^{99}} = 0$, since $\alpha^{99.7} = 1$ and $\alpha^{99} \neq 1$.

6. Find the roots of the equation $z^n = (z + 1)^n$, where n is a positive integer > 1. Show that the points which represent them in the z-plane are collinear.

Let $w = \frac{z+1}{z}$. Then $z = \frac{1}{w-1}$. $z^n = (z + 1)^n \Rightarrow w^n = 1$.

Therefore $w = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$, where $k = 0, 1, 2, \ldots, n - 1$.

Therefore $z = \frac{1}{\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} - 1}$, where $k = 1, 2, \ldots, n - 1$

$$= \frac{1}{2i \sin \frac{k\pi}{n} (\cos \frac{k\pi}{n} + i \sin \frac{k\pi}{n})} = \frac{\cos \frac{k\pi}{n} - i \sin \frac{k\pi}{n}}{2i \sin \frac{k\pi}{n}}$$

$= -\frac{1}{2} - \frac{i}{2} \cot \frac{k\pi}{n}$, where $k = 1, 2, \ldots, n - 1$.

The points represented by $n - 1$ roots lie on the line $x = -\frac{1}{2}$ which is parallel to the imaginary axis.

7. Find the product of all the values of $(1 + i)^{\frac{4}{5}}$.

$(1 + i)^{\frac{4}{5}} = [\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})]^{\frac{4}{5}} = [4(\cos \pi + i \sin \pi)]^{\frac{1}{5}}$.

There are five distinct values. They are

$\sqrt[5]{4}[\cos \frac{2k\pi+\pi}{5} + i \sin \frac{2k\pi+\pi}{5}]$, where $k = 0, 1, \ldots, 4$

$= \sqrt[5]{4}[\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}](\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5})^k$, where $k = 0, 1, \ldots, 4$.

The product $= 4(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5})^5 (\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5})^{0+1+2+3+4} = -4$.

8. Prove that $\sqrt{i} + \sqrt{-i} = \sqrt{2}$.

$i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$, $-i = \cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2})$.

The square roots of i are $\cos \frac{2k\pi+\frac{\pi}{2}}{2} + i \sin \frac{2k\pi+\frac{\pi}{2}}{2}$, where $k = 0, 1$.

The principal square root is $\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$ (corresponding to $k = 0$).
That is, $\sqrt{i} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$.

The square roots of $-i$ are $\cos \frac{2k\pi-\frac{\pi}{2}}{2} + i \sin \frac{2k\pi-\frac{\pi}{2}}{2}$, where $k = 0, 1$.

The principal square root is $\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})$(corresponding to $k = 0$). That is, $\sqrt{-i} = \cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}) = \frac{1-i}{\sqrt{2}}$.

Therefore $\sqrt{i} + \sqrt{-i} = \frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}} = \sqrt{2}$.

9. Prove that $\sqrt[n]{i} + \sqrt[n]{-i} = 2 \cos \frac{\pi}{2n}$, where n is a positive integer > 1 and $\sqrt[n]{z}$ is the principal nth root of z.

$i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$. $-i = \cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2})$.

$\sqrt[n]{i} = \cos \frac{\pi}{2n} + i \sin \frac{\pi}{2n}$. $\sqrt[n]{-i} = \cos(-\frac{\pi}{2n}) + i \sin(-\frac{\pi}{2n}) = \cos \frac{\pi}{2n} - i \sin \frac{\pi}{2n}$.

Therefore $\sqrt[n]{i} + \sqrt[n]{-i} = 2 \cos \frac{\pi}{2n}$.

2.11. Some applications of De Moivre's theorem.

1. Expansion of $\cos n\theta, \sin n\theta, \tan n\theta$ when n is a positive integer and θ is real.

When n is a positive integer, by De Moivre's theorem

$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$

$= \cos^n \theta + {}^nC_1 \cos^{n-1} \theta i \sin \theta + {}^nC_2 \cos^{n-2} \theta i^2 \sin^2 \theta + {}^nC_3 \cos^{n-3} \theta i^3 \sin^3 \theta + \cdots + i^n \sin^n \theta$

$= (\cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + {}^nC_4 \cos^{n-4} \theta \sin^4 \theta + \cdots) + i({}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3} \theta \sin^3 \theta + \cdots)$.

Consequently, $\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \cdots$ and

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots$

If n be even, the last term in $\cos n\theta$ and $\sin n\theta$ are $(-1)^{n/2}\sin^n\theta$ and $(-1)^{(n-2)/2}n\cos\theta\sin^{n-1}\theta$ respectively.

If n be odd, those are $(-1)^{(n-1)/2}n\cos\theta\sin^{n-1}\theta$ and $(-1)^{(n-1)/2}n\sin^n\theta$ respectively.

Hence $\tan n\theta = \dfrac{\sin n\theta}{\cos n\theta}$

$$= \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots}$$

$$= \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + \ldots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots}.$$

If n be even, the last term in the numerator is $(-1)^{(n-2)/2}n\tan^{n-1}\theta$ and that in the denominator is $(-1)^{n/2}\tan^n\theta$.

If n be odd the last terms are $(-1)^{(n-1)/2}\tan^n\theta$ and $(-1)^{(n-1)/2}n\tan^{n-1}\theta$ respectively.

2. Expression for $\tan(\theta_1+\theta_2+\cdots\theta_n)$, where $\theta_1,\theta_2,\ldots,\theta_n$ are real.

We have $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)$.

Therefore $\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)$
$= \cos\theta_1(1+i\tan\theta_1)\cos\theta_2(1+i\tan\theta_2)\ldots\cos\theta_n(1+i\tan\theta_n)$
$= \cos\theta_1\cos\theta_2\cdots\cos\theta_n(1 + is_1 + i^2s_2 + \cdots + i^ns_n)$, where
$s_1 = \Sigma\tan\theta_1, s_2 = \Sigma\tan\theta_1\tan\theta_2, s_3 = \Sigma\tan\theta_1\tan\theta_2\tan\theta_3, \ldots$

We have $\cos(\theta_1+\theta_2+\cdots+\theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1 - s_2 + s_4 - \ldots)$ and
$\sin(\theta_1+\theta_2+\cdots+\theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(s_1 - s_3 + s_5 - \ldots)$.

Hence $\tan(\theta_1+\theta_2+\cdots+\theta_n) = \dfrac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - \ldots}$.

If n be odd, the last term in the numerator is $(-1)^{(n-1)/2}s_n$ and that in the denominator is $(-1)^{(n-1)/2}s_{n-1}$.

If n be even, those are $(-1)^{(n-2)/2}s_{n-1}$ and $(-1)^{n/2}s_n$ respectivly.

3. Expansion of $\cos^n\theta$ in a series of multiples of θ when n is a positive integer and θ is real.

Let $x = \cos\theta + i\sin\theta$.
Then $x^n = \cos n\theta + i\sin n\theta, x^{-n} = \cos n\theta - i\sin n\theta$.
$x^n + x^{-n} = 2\cos n\theta, x^n - x^{-n} = 2i\sin n\theta$.

We have $(2\cos\theta)^n = (x + \frac{1}{x})^n$
$= x^n + {}^nC_1x^{n-1}\cdot\frac{1}{x} + {}^nC_2x^{n-2}\cdot\frac{1}{x^2} + \cdots + {}^nC_{n-1}x\cdot\frac{1}{x^{n-1}} + \frac{1}{x^n}$
or, $(2^n\cos^n\theta) = (x^n + \frac{1}{x^n}) + {}^nC_1(x^{n-2} + \frac{1}{x^{n-2}}) + \cdots$
$= 2\cos n\theta + {}^nC_1.2\cos(n-2)\theta + {}^nC_2.2\cos(n-4)\theta + \ldots$

The expansion of $(x + \frac{1}{x})^n$ contains $n+1$ terms. If n be even, there is a middle term which is free from x. If n be odd, there are two middle terms, one containing x and the other containing $\frac{1}{x}$.

The last term in the expansion of $(2\cos\theta)^n$ is a constant if n be even and is a term containing $\cos\theta$ if n be odd.

4. Expansion of $\sin^n\theta$ in a series of cosines or sines of multiplies θ according as n is an even or odd positive integer.

Let $x = \cos\theta + i\sin\theta$.

Then $x + \frac{1}{x} = 2\cos\theta, x - \frac{1}{x} = 2i\sin\theta$.

$x^n + \frac{1}{x^n} = 2\cos n\theta, x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Case 1. Let n be even.

We have $(2i\sin\theta)^n = (x - \frac{1}{x})^n$
$= x^n - {}^nC_1x^{n-2} + {}^nC_2x^{n-4} - \cdots - {}^nC_{n-1}\cdot\frac{1}{x^{n-2}} + \frac{1}{x^n}$
$= (x^n + \frac{1}{x^n}) - {}^nC_1(x^{n-2} + \frac{1}{x^{n-2}}) + \cdots$
$= 2\cos n\theta - {}^nC_1.2\cos(n-2)\theta + \cdots$

Since n is even, the expansion of $(x - \frac{1}{x})^n$ contains a middle term which is free from x.

The last term in the expansion of $(2i\sin\theta)^n$ is a constant.

Case 2. Let n be odd.

$(2i\sin\theta)^n = (x - \frac{1}{x})^n$
$= x^n - {}^nC_1x^{n-2} + {}^nC_2x^{n-4} - \cdots + {}^nC_{n-1}\cdot\frac{1}{x^{n-2}} - \frac{1}{x^n}$

or, $2^ni(-1)^{(n-1)/2}\sin^n\theta = (x^n - \frac{1}{x^n}) - {}^nC_1(x^{n-2} - \frac{1}{x^{n-2}}) + \cdots$
$= 2i\sin n\theta - {}^nC_1\,2i\sin(n-2)\theta + \cdots$

Therefore $2^{n-1}(-1)^{(n-1)/2}\sin^n\theta = \sin n\theta - {}^nC_1\sin(n-2)\theta + {}^nC_2\sin(n-4)\theta - \cdots$

Since n is odd, the expansion of $(x - \frac{1}{x})^n$ contains two middle terms of opposite signs, one containing x and the other containing $\frac{1}{x}$.

The last term in the expansion of $2^{n-1}(-1)^{(n-1)/2}\sin^n\theta$ is $(-1)^{\frac{n-1}{2}}\cdot{}^nC_{\frac{n-1}{2}}\sin\theta$.

Note. The methods discussed in **3** and **4** can also be used to express $\sin^m\theta\cos^n\theta$ in a series of sines or cosines of multiples of θ, when m, n are positive integers.

Worked Examples.

1. Use De Moivre's theorem to prove tan $5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

We have $\cos 5\theta + i \sin 5\theta = (\cos\theta + i\sin\theta)^5$

$$\begin{aligned} &= \cos 5\theta + 5c_1 \cos^4\theta i \sin\theta + 5c_2 \cos^3\theta i^2 \sin^2\theta \\ &+ 5c_3 \cos^2\theta i^3 \sin^3\theta + \cdots + i^5 \sin^5\theta \\ &= (\cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta) \\ &+ i(5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta). \end{aligned}$$

Therefore $\cos 5\theta = \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta)$,
$\sin 5\theta = 5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta$.

Hence $\tan 5\theta = \frac{5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta}$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

2. Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

From Example 1, $\cos 5\theta = \cos^5\theta - 10\cos^3 \sin^2\theta + 5\cos\theta \sin^4\theta$

$$\begin{aligned} &= t^5 - 10t^3(1-t^2) + 5t(1-t^2)^2, \text{ where } t = \cos\theta \\ &= t^5 - 10t^3(1-t^2) + 5t(1 - 2t^2 + t^4) \\ &= 16t^5 - 20t^3 + 5t \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta. \end{aligned}$$

3. Expand $\cos^7\theta$ in a series of cosines of multiples of θ.

Let $x = \cos\theta + i\sin\theta$. Then $x^n + \frac{1}{x^n} = 2\cos n\theta$.

$$\begin{aligned} &(2\cos\theta)^7 = (x + \tfrac{1}{x})^7 \\ &= x^7 + 7x^5 + 21x^3 + 35x + 35.\tfrac{1}{x} + 21.\tfrac{1}{x^3} + 7.\tfrac{1}{x^5} + \tfrac{1}{x^7} \\ &= (x^7 + \tfrac{1}{x^7}) + 7(x^5 + \tfrac{1}{x^5}) + 21(x^3 + \tfrac{1}{x^3}) + 35(x + \tfrac{1}{x}) \\ &= 2\cos 7\theta + 7.2\cos 5\theta + 21.2\cos 3\theta + 35.2\cos\theta. \end{aligned}$$

Therefore $\cos^7\theta = \frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$.

4. Expand $\sin^4\theta \cos^2\theta$ in a series of cosines of multiples of θ.

Let $x = \cos\theta + i\sin\theta$. Then $x + \frac{1}{x} = 2\cos\theta, x - \frac{1}{x} = 2i\sin\theta$,
$x^n + \frac{1}{x^n} = 2\cos n\theta, \quad x^n - \frac{1}{x^n} = 2i\sin n\theta$.

$$\begin{aligned} &(2i\sin\theta)^4(2\cos\theta)^2 = (x - \tfrac{1}{x})^4(x + \tfrac{1}{x})^2 = (x^2 - \tfrac{1}{x^2})^2(x - \tfrac{1}{x})^2 \\ &= (x^4 - 2 + \tfrac{1}{x^4})(x^2 - 2 + \tfrac{1}{x^2}) \\ &= (x^6 + \tfrac{1}{x^6}) - 2(x^4 + \tfrac{1}{x^4}) - (x^2 + \tfrac{1}{x^2}) + 4 \\ &= 2\cos 6\theta - 4\cos 4\theta - 2\cos 2\theta + 4. \end{aligned}$$

Therefore $\sin^4\theta \cos^2\theta = \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)$.

Exercises 2A

1. z_1 and z_2 are complex numbers. Prove that

(i) $z_1\overline{z_2} + \overline{z_1}z_2 \le 2\,|z_1|\,|z_2|$,

(ii) $|1 - \overline{z_1}z_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$,

(iii) $|1 + \overline{z_1}z_2|^2 + |z_1 - z_2|^2 = (1 + |z_1|^2)(1 + |z_2|^2)$.

2. z_1 and z_2 are two non-zero complex numbers. Prove that

(i) $|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}| \le 2$,

(ii) $2|z_1 + z_2| \ge (|z_1| + |z_2|)\,|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}|$.

3. Prove that for a complex number z, $|z| \ge \frac{1}{\sqrt{2}}(|Rlz| + |Imz|)$.

[Hint. Let $z = a + ib$. Use the inequality $\frac{|a|^2 + |b|^2}{2} \ge (\frac{|a| + |b|}{2})^2$.]

4. z_1, z_2 are complex numbers and $w = \sqrt{z_1 z_2}$. Prove that

$|z_1| + |z_2| = |\frac{z_1 + z_2}{2} + w| + |\frac{z_1 + z_2}{2} - w|$.

[Hint. $(|\frac{z_1 + z_2}{2} + w| + |\frac{z_1 + z_2}{2} - w|)^2 = |\frac{z_1 + z_2}{2} + w|^2 + |\frac{z_1 + z_2}{2} - w|^2 + 2|\frac{z_1 - z_2}{2}|^2 = 2|\frac{z_1 + z_2}{2}|^2 + 2|w|^2 + 2|\frac{z_1 - z_2}{2}|^2$, since $|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2$.]

[Note that $\sqrt{z_1 z_2} \ne \sqrt{z_1}.\sqrt{z_2}$, in general.]

5. z_1, z_2 are complex numbers such that $|z_1 - 3z_2| = |3 - z_1\bar{z}_2|$ and $|z_2| \ne 1$. Prove that $|z_1| = 3$.

6. z_1, z_2 are complex numbers such that $z_1 + z_2$ and $z_1 z_2$ are both real. Prove that either z_1 and z_2 are purely real, or $z_1 = \overline{z_2}$.

7. z_1, z_2 are complex numbers such that $\frac{z_1}{z_2}$ is real. Prove that the points representing z_1 and z_2 in the complex plane are collinear with the origin.

8. z is a variable complex number such that the ratio $\frac{z - i}{z + 1}$ is purely imaginary. Show that the point z lies on a circle in the complex plane.

9. z is a variable complex number such that an amplitude of $\frac{z - i}{z + 1}$ is $\frac{\pi}{4}$. Show that the point z lies on a circle in the complex plane.

10. z is a variable complex number such that mod $\frac{z - i}{z + 1} = k$. Show that the point z lies on a circle in the complex plane if $k \ne 1$ and z lies on a straight line if $k = 1$.

11. z is a variable complex number such that $|z| = 2$. Show that the point $z + \frac{1}{z}$ lies on an ellipse of eccentricity $\frac{4}{5}$ in the complex plane.

12. Find the complex number z with the least positive argument that satisfies the condition $|z - 5i| \le 4$.

13. z is a variable complex number such that $|\, z - \frac{10}{z} \,| = 3$. Find the greatest and the least value of $|\, z \,|$.

14. Three complex numbers z_1, z_2, z_3 are such that $z_1 + z_2 + z_3 = 0$ and $|\, z_1 \,| = |\, z_2 \,| = |\, z_3 \,|$. Prove that they represent the vertices of an equilateral triangle in the complex plane.

15. If z_1, z_2, z_3 be three complex numbers such that ${z_1}^2 + {z_2}^2 + {z_3}^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$, prove that
$|\, z_1 - z_2 \,| = |\, z_2 - z_3 \,| = |\, z_3 - z_1 \,|$.

16. Prove that three complex numbers z_1, z_2, z_3 will represent the vertices of an equilateral triangle in the complex plane if and only if $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$.

[**Hint.** Let $z_2 - z_3 = \alpha$, etc. Then $\alpha + \beta + \gamma = 0$ and therefore $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$.
$|\alpha| = |\beta| = |\gamma| \Rightarrow \alpha\bar{\alpha} = \beta\bar{\beta} = \gamma\bar{\gamma} = k$, say. Then $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0$.
Conversely, $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0 \Rightarrow \alpha^2 = \beta\gamma \Rightarrow \alpha^3 = \alpha\beta\gamma \Rightarrow |\alpha|^3 = |\alpha\beta\gamma|$.]

17. The roots of the equation $z^2 + pz + q = 0$, where p and q are complex numbers, are represented by the points A, B on the complex plane. If $OA = OB$ and $\angle AOB = 2\beta$, where O is the origin, prove that
$$p^2 = 4q\cos^2\beta.$$

18. (i) Two complex numbers z_1, z_2 are such that $|\, z_1 \,| = |\, z_2 \,|$ and amp z_1, amp z_2 differ by π. Prove that $z_1 + z_2 = 0$.

(ii) Two complex numbers z_1, z_2 are such that $|\, z_1 + z_2 \,| = |\, z_1 - z_2 \,|$. Prove that amp z_1 and amp z_2 differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

19. Find mod z and amp z (principal amplitude) where

(i) $z = \frac{1+\sqrt{3}i}{1+i}$ (ii) $z = \frac{-1+i}{1-\sqrt{3}i}$

(iii) $z = 1 + i\tan\theta, \quad \frac{\pi}{2} < \theta < \pi$

(iv) $z = 1 + i\cot\theta, \quad 0 < \theta < \pi$

(v) $z = \frac{1+\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}, \quad 0 < \theta < \pi$

(vi) $z = \frac{1+\sin\theta-i\cos\theta}{1+\sin\theta+i\cos\theta}, \quad 0 < \theta < \pi$

(vii) $z = 1 + \cos 2\theta + i\sin 2\theta, \quad \frac{\pi}{2} < \theta < \pi$

(viii) $z = 1 + \cos 2\theta - i\sin 2\theta, \quad \frac{\pi}{2} < \theta < \pi$

(ix) $z = 1 - \cos\theta(\cos\theta + i\sin\theta), \quad 0 < \theta < \pi$

(x) $z = 1 - \sin\theta(\sin\theta + i\cos\theta), \quad \frac{\pi}{2} < \theta < \pi.$

20. If $|z| = 1$ and amp $z = \theta$ $(0 < \theta < \pi)$, find the modulus and the principal amplitude of

(i) $\frac{1-z}{1+z}$, (ii) $\frac{2}{1-z}$, (iii) $\frac{2}{1+z}$, (iv) $\frac{2}{1-z^2}$.

21. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, prove that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$,

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$,

(iii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$,

(iv) $\cos(\beta + \gamma) + \cos(\gamma + \alpha) + \cos(\alpha + \beta) = 0$,

(v) $\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0$.

22. If $x = \cos\alpha + i\sin\alpha, y = \cos\beta + i\sin\beta, z = \cos\gamma + i\sin\gamma$ and $x + y + z = xyz$, prove that
$$\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -1.$$

23. If n be an integer, prove that
$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right).$$

24. If $z = \cos\theta + i\sin\theta$ and m is a positive integer, prove that

(i) $\frac{z^{2m}-1}{z^{2m}+1} = i\tan m\theta$ (ii) $(1+z)^m + (1+\frac{1}{z})^m = 2^{m+1}\cos^m\frac{\theta}{2}\cos\frac{m\theta}{2}$.

25. If $2\cos\theta = x + \frac{1}{x}$ and θ is real, prove that
$$2\cos n\theta = x^n + \frac{1}{x^n}, n \text{ being an integer.}$$

26. If $2\cos\theta = t$ prove that
$$\frac{1+\cos 7\theta}{1+\cos\theta} = (t^3 - t^2 - 2t + 1)^2.$$

[**Hint.** Let $x = \cos\theta + i\sin\theta$. Then $x + \frac{1}{x} = t$ and $\frac{1+\cos 7\theta}{1+\cos\theta} = \frac{1}{x^6}(\frac{x^7+1}{x+1})^2$.]

27. If $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and if p is prime to n, prove that
$$1 + \alpha^p + \alpha^{2p} + \cdots + \alpha^{(n-1)p} = 0.$$

28. If n be a positive integer and $(1+z)^n = p_0 + p_1z + p_2z^2 + \cdots + p_nz^n$, prove that

(i) $p_0 - p_2 + p_4 - \cdots = 2^{n/2}\cos\frac{n\pi}{4}$,

(ii) $p_1 - p_3 + p_5 - \cdots = 2^{n/2}\sin\frac{n\pi}{4}$,

(iii) $p_0 + p_4 + p_8 + \cdots = 2^{n-2} + 2^{(n-2)/2}\cos\frac{n\pi}{4}$,

(iv) $p_0 + p_3 + p_6 + \cdots = \frac{2}{3}(2^{n-1} + \cos\frac{n\pi}{3})$,

(v) $p_1 + p_4 + p_7 + \cdots = \frac{2}{3}(2^{n-1} + \cos\frac{(n-2)\pi}{3})$,

(vi) $p_2 + p_5 + p_8 + \cdots = \frac{2}{3}(2^{n-1} + \cos\frac{(n+2)\pi}{3})$.

[Hint. (iv)-(vi) Put $z = 1, z = \omega, z = \omega^2$ successively.
$3(p_0 + p_3 + p_6 + \cdots) = 2^n + (1+\omega)^n + (1+\omega^2)^n$.
$3(p_1 + p_4 + p_7 + \cdots) = 2^n + \omega^2(1+\omega)^n + \omega(1+\omega^2)^n$.
$3(p_2 + p_5 + p_8 + \cdots) = 2^n + \omega(1+\omega)^n + \omega^2(1+\omega^2)^n$.]

29. If α, β are the roots of the equation $t^2 - 2t + 5 = 0$ and n is a positive integer, prove that

$$\frac{(\alpha+\alpha)^n-(\alpha+\beta)^n}{\alpha-\beta} = 2^{n-1}\sin n\phi \operatorname{cosec}^n \phi,$$ where α is a real number satisfying $\frac{\alpha+1}{2} = \cot \phi$.

30. If α, β are the roots of the equation $t^2 + 2t + 4 = 0$ and m is a positive integer, prove that

$$\alpha^m + \beta^m = 2^{m+1}\cos\tfrac{2m\pi}{3}.$$

31. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $t^4 + t^2 + 1 = 0$ and n is a positive integer, prove that

(i) $\alpha^{2n} + \beta^{2n} + \gamma^{2n} + \delta^{2n} = 4\cos\frac{2n\pi}{3}$,

(ii) $\alpha^{2n+1} + \beta^{2n+1} + \gamma^{2n+1} + \delta^{2n+1} = 0$.

32. If m be a positive integer and s_m be the sum of mth power of the roots of the equation $x^6 + x^3 + 1 = 0$, prove that $s_{3m} = 6\cos\frac{2m\pi}{3}$.

33. Solve the equations

(i) $x^3 + 8 = 0$, (ii) $x^5 + x^3 + x^2 + 1 = 0$,

(iii) $x^4 + 2x^2 + 4 = 0$, (iv) $x^6 + x^3 + 1 = 0$,

(v) $x^4 + (x-1)^4 = 0$, (vi) $x^8 = (x+1)^8$,

(vii) $x^6 + 2x^3 + 2 = 0$, (viii) $(x+1)^6 = (x-1)^6$.

34. Show that the solutions of the equation $(1+x)^{2n} + (1-x)^{2n} = 0$ are $x = \pm i\tan\frac{(2r-1)\pi}{4n}$, $r = 1, 2, \ldots, n$.

35. Show that the solutions of the equation $(1+x)^{2n+1} - (1-x)^{2n+1} = 0$ are $x = 0, \pm i\tan\frac{r\pi}{2n+1}$, $r = 1, 2, \ldots, n$.

36. Show that the solutions of the equation $(1+x)^n - (1-x)^n = 0$ are $x = i\tan\frac{r\pi}{n}$, where

$r = 0, 1, 2, \ldots, n-1$, if n be odd

$= 0, 1, \ldots, \frac{n}{2} - 1, \frac{n}{2} + 1, \ldots, n-1$, if n be even.

37. Find all values of

(i) $(-1)^{1/4}$, (ii) $i^{1/3}$, (iii) $(-i)^{1/4}$, (iv) $(-i)^{3/4}$.

38. (i) Show that the product of all the values of $(1+\sqrt{3}i)^{\frac{3}{4}}$ is 8.

(ii) Show that the product of all the values of $(\sqrt{3}+i)^{\frac{3}{4}}$ is $8i$.

(iii) Show that the sum of the squares of all the values of $(\sqrt{3}+i)^{\frac{3}{4}}$ is 0.

39. In a triangle ABC, prove that

$$a^3\cos 3B + 3a^2b\cos(2B - A) + 3ab^2\cos(B - 2A) + b^3\cos 3A = c^3,$$

where a, b, c, A, B, C have their usual meanings.

40. Prove that

(i) $x^n - 1 = (x^2-1)\prod_{k=1}^{(n-2)/2}[x^2 - 2x\cos\frac{2k\pi}{n} + 1]$, if n be an even positive integer.

Deduce that $\sin\frac{\pi}{32}\sin\frac{2\pi}{32}\sin\frac{3\pi}{32}\cdots\sin\frac{15\pi}{32} = \frac{1}{2^{15}}$.

(ii) $x^n - 1 = (x-1)\prod_{k=1}^{(n-1)/2}[x^2 - 2x\cos\frac{2k\pi}{n} + 1]$, if n be an odd positive integer.

Deduce that $\sin\frac{\pi}{25}\sin\frac{2\pi}{25}\sin\frac{3\pi}{25}\cdots\sin\frac{12\pi}{25} = \frac{5}{2^{12}}$.

41. Prove that

(i) $x^n + 1 = \prod_{k=0}^{(n-2)/2}[x^2 - 2x\cos\frac{(2k+1)\pi}{n} + 1]$, if n be an even positive integer.

Deduce that $\sin\frac{\pi}{16}\sin\frac{3\pi}{16}\sin\frac{5\pi}{16}\sin\frac{7\pi}{16} = \frac{1}{8\sqrt{2}}$.

(ii) $x^n + 1 = (x+1)\prod_{k=0}^{(n-3)/2}[x^2 - 2x\cos\frac{(2k+1)\pi}{n} + 1]$, if n be an odd positive integer.

Deduce that (i) $\sin\frac{\pi}{18}\sin\frac{3\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18} = \frac{1}{16}$

(ii) $\cos\frac{\pi}{18}\cos\frac{3\pi}{18}\cos\frac{5\pi}{18}\cos\frac{7\pi}{18} = \frac{3}{16}$.

42. Prove that

(i) $x^8 + y^8 = \Pi(x^2 - 2xy\cos\frac{r\pi}{8} + y^2)$, $r = 1, 3, 5, 7$.

(ii) $x^8 - 2x^4\cos 4\theta + 1 = \prod_{r=0}^{3}[x^2 - 2x\cos(\theta + \frac{r\pi}{2}) + 1]$

(iii) $x^4 + \frac{1}{x^4} - 2\cos 4\theta = \prod_{r=0}^{3}[x + \frac{1}{x} - 2\cos(\theta + \frac{r\pi}{2})]$

[Hint. Divide both sides of (ii) by x^4.]

(iv) $\cos 4\phi - \cos 4\theta = 8\prod_{r=0}^{3}[\cos\phi - \cos(\theta + \frac{r\pi}{2})]$.

[Hint. Let $x = \cos\phi + i\sin\phi$ in (iii).]

43. Prove that

(i) $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$,

(ii) $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$,

(iii) $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$,

(iv) $\cos^8\theta = \frac{1}{128}(\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35)$,

(v) $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$,

(vi) $2^8\sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta$.

2.12. Exponential function.

When x is real the infinite series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

converges for all x and the sum is denoted by e^x. e^x is a function of a real variable x defined for all x. This is called the *exponential function* of a real variable x.

For a complex variable $z = x + iy$, the exponential function of z, written as $\exp z$, is defined by $\exp(x + iy) = e^x(\cos y + i \sin y)$.

This definition agrees with the real exponential function when z is purely real.

When z is purely real, $y = 0$ and $\exp z = e^x(\cos 0 + i \sin 0)$, i.e., $\exp x = e^x$.

When z is purely imaginary, $x = 0$ and $\exp z = (\cos y + i \sin y)$, i.e., $\exp(iy) = \cos y + i \sin y$.

Since $e^x > 0$ for all real x, $e^x(\cos y + i \sin y)$ represents a complex number in polar form, e^x being the modulus and y being an amplitude of $\exp z$.

Since $e^x \neq 0$ for any real number x, $\exp z$ is a non-zero complex number for any complex number z.

Let $u + iv$ be a *non-zero* complex number and let its polar representation be $r(\cos \theta + i \sin \theta)$. Since r is positive, $\log r$ is real and r can be expressed as $r = e^{\log r}$.

$$\begin{aligned} \text{Therefore } u + iv &= e^{\log r}(\cos \theta + i \sin \theta) \\ &= \exp(\log r + i\theta). \end{aligned}$$

Thus when $u + iv$ is a given *non-zero* complex number, there exists a complex number $z = \log r + i\theta$ such that $\exp z = u + iv$. This means that the range of the exponential function of z is the entire complex plane excluding the origin.

Properties.

1. $\exp z_1 . \exp z_2 = \exp(z_1 + z_2)$, where z_1, z_2 are complex numbers.

Proof. Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$.

Then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.

$\exp z_1 = e^{x_1}(\cos y_1 + i \sin y_1)$, $\exp z_2 = e^{x_2}(\cos y_2 + i \sin y_2)$.

$$\begin{aligned} \exp z_1 . \exp z_2 &= e^{x_1}(\cos y_1 + i \sin y_1)e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2}[\cos(y_1 + y_2) + i \sin(y_1 + y_2)] \\ &= \exp[(x_1 + x_2) + i(y_1 + y_2)] \\ &= \exp(z_1 + z_2). \end{aligned}$$

2. $\frac{\exp z_1}{\exp z_2} = \exp(z_1 - z_2)$.

Proof. Since $\exp z_2$ is a non-zero complex number, $\frac{\exp z_1}{\exp z_2}$ is defined.

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$.

Then $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$.

$\exp z_1 = e^{x_1}(\cos y_1 + i \sin y_1)$, $\exp z_2 = e^{x_2}(\cos y_2 + i \sin y_2)$.

$$\begin{aligned} \frac{\exp z_1}{\exp z_2} &= e^{x_1 - x_2}\frac{\cos y_1 + i \sin y_1}{\cos y_2 + i \sin y_2} \\ &= e^{x_1 - x_2}[\cos(y_1 - y_2) + i \sin(y_1 - y_2)] \\ &= \exp[(x_1 - x_2) + i(y_1 - y_2)] \\ &= \exp(z_1 - z_2). \end{aligned}$$

Corollary. $\frac{1}{\exp z} = \exp(-z)$.

This follows from the property since $\exp(0) = 1$

3. If n be an integer, $(\exp z)^n = \exp(nz)$.

This follows from the property **1** and the relation $(\exp z)^{-1} = \exp(-z)$.

Note. If θ be real and n be an integer, it follows that $(\exp i\theta)^n = \exp in\theta$, i.e., $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. This is De Moivre's theorem.

4. If n be an fraction say p/q, $(\exp z)^n$ has q distinct values but $\exp(nz)$ is unique. In this case, $\exp(nz)$ is one of the values of $(\exp z)^n$.

Note. If θ be real and n be a fraction, it follows that $(\exp in\theta)$ is one of the values of $(\exp i\theta)^n$, i.e., $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

5. If n be an integer, $\exp(z + 2n\pi i) = \exp z$.

This follows from the property **1** and the relation $\exp(2n\pi i) = 1$.

This states that exponential function is periodic with period $2\pi i$.

Note. A complex function f is said to be a *periodic function* on its domain $D \subset \mathbb{C}$ if there exists a non-zero constant w such that for all integers n, $f(z + nw) = f(z)$ (*) holds for all $z \in D$. If no submultiple of w satisfies the relation (*), then w is said to be the *period* of f.

Let w be a non-zero complex number. Then there always exists a complex number z such that $\exp z = w$.

By the property **5**, $\exp z = w \Rightarrow \exp(z + 2n\pi i) = w$, where n is an integer.

Thus for a non-zero complex number w there exist infinitely many complex numbers z such that $\exp z = w$.

Worked Examples.

1. If α, β be real, find the sum

(i) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots$ to n terms;

(ii) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots$ to n terms.

[Here the angles are in arithmetic progression with the first term α and the common difference β]

Let $c_n = \cos\alpha + \cos(\alpha+\beta) + \cdots + \cos(\alpha+\overline{n-1}\beta)$,
$s_n = \sin\alpha + \sin(\alpha+\beta) + \cdots + \sin(\alpha+\overline{n-1}\beta)$.

Then $c_n + is_n = \exp i\alpha + \exp i(\alpha+\beta) + \cdots + \exp i(\alpha+\overline{n-1}\beta)$

$= (\exp i\alpha)[1 + (\exp i\beta) + (\exp i\beta)^2 + \cdots + (\exp i\beta)^{n-1}]$

$= (\exp i\alpha)[\frac{1-(\exp i\beta)^n}{1-\exp i\beta}]$

$= (\cos\alpha + i\sin\alpha)[\frac{1-\cos n\beta - i\sin n\beta}{1-\cos\beta - i\sin\beta}]$

$= (\cos\alpha + i\sin\alpha)[\frac{(1-\cos n\beta - i\sin n\beta)(1-\cos\beta+i\sin\beta)}{(1-\cos\beta)^2+\sin^2\beta}]$

$= (\cos\alpha + i\sin\alpha)[2\sin^2\frac{n\beta}{2} - i2\sin\frac{n\beta}{2}\cos\frac{n\beta}{2}][\frac{2\sin^2\frac{\beta}{2}+i2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{4\sin^2\frac{\beta}{2}}]$

$= (\cos\alpha + i\sin\alpha)(-2i\sin\frac{n\beta}{2})(\cos\frac{n\beta}{2} + i\sin\frac{n\beta}{2})(2i\sin\frac{\beta}{2})$
$[\frac{(\cos\frac{\beta}{2} - i\sin\frac{\beta}{2})}{4\sin^2\frac{\beta}{2}}]$

$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}[\cos(\alpha + \frac{\overline{n-1}\beta}{2}) + i\sin(\alpha + \frac{\overline{n-1}\beta}{2})]$.

Therefore $c_n = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}[\cos(\alpha + \frac{\overline{n-1}\beta}{2})]$ and $s_n = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}[\sin(\alpha + \frac{\overline{n-1}\beta}{2})]$.

2. Find all complex numbers z such that $\exp z = -1$.

Let $z = x + iy$.
Then $\exp z = -1$ implies $e^x(\cos y + i \sin y) = -1$.
Therefore $e^x \cos y = -1, e^x \sin y = 0$.

We have $e^x = 1$ and $\cos y = -1, \sin y = 0$.
$e^x = 1 \Rightarrow x = 0$,
$\cos y = -1$ and $\sin y = 0 \Rightarrow y = (2n+1)\pi$, where n is an integer.
Therefore $z = (2n+1)\pi i$.

3. Find all complex numbers z such that $\exp(2z+1) = i$.

Let $z = x + iy$. Then $2z + 1 = (2x+1) + i2y$.
$\exp(2z+1) = i$ implies $e^{2x+1}(\cos 2y + i \sin 2y) = i$.

Therefore $e^{2x+1}\cos 2y = 0, e^{2x+1}\sin 2y = 1$.

We have $e^{2x+1} = 1$ and $\cos 2y = 0, \sin 2y = 1$.
$e^{2x+1} = 1 \Rightarrow x = -\frac{1}{2}$,
$\cos 2y = 0$ and $\sin 2y = 1 \Rightarrow y = (4n+1)\frac{\pi}{4}$, where n is an integer.
Therefore $z = -\frac{1}{2} + (4n+1)\frac{\pi}{4}i$.

4. Solve $\exp z = 1 + \sqrt{3}i$.

Let $z = x + iy$.
Then $\exp z = 1 + \sqrt{3}i$ implies $e^x(\cos y + i \sin y) = 1 + \sqrt{3}i$.
Therefore $e^x \cos y = 1, e^x \sin y = \sqrt{3}$.

We have $e^x = 2$ and $\cos y = \frac{1}{2}, \sin y = \frac{\sqrt{3}}{2}$.
These determine $x = \log 2$ and $y = 2n\pi + \frac{\pi}{3}$, where n is an integer.
Therefore $z = \log 2 + (2n\pi + \frac{\pi}{3})i$.

2.13. Logarithmic function.

Let z be a *non-zero* complex number. Then there always exists a complex number w such that $\exp w = z$.

w is said to be a *logarithm* of z.

Again $\exp w = \exp(w + 2n\pi i)$, where n is an integer. This shows that if w is a logarithm of z, then $w + 2n\pi i$ is also a logarithm of z.

This means that "logarithm of z" is a many-valued function of z. This is denoted by $\text{Log } z = w + 2n\pi i$.

Of the many values of logarithm of z, a particular one is called the *principal logarithm* of z and is denoted by $\log z$.

Since z is a *non-zero* complex number, z has a polar representation.
Let $z = r(\cos\theta + i \sin\theta), -\pi < \theta \le \pi$ (a polar form with amp z).

Let $w = u + iv$ be a logarithm of z. Then $\exp w = z$. This gives
$e^u(\cos v + i \sin v) = r(\cos\theta + i \sin\theta)$.
Therefore $e^u \cos v = r\cos\theta, e^u \sin v = r\sin\theta$.
We have $e^u = r$ and $\cos v = \cos\theta, \sin v = \sin\theta$.

These determine $u = \log r$ and $v = \theta + 2n\pi$, where n is an integer.
Therefore $w = \log r + i(\theta + 2n\pi), -\pi < \theta \le \pi$

i.e., $\text{Log } z = \log r + i(\theta + 2n\pi)$
$= \log|z| + i(\arg z + 2n\pi)$.

The principal logarithm of z, denoted by $\log z$, corresponds to $n = 0$.
Therefore $\log z = \log r + i\theta, -\pi < \theta \le \pi$
$= \log|z| + i \arg z$.

Note 1. Log z has been expressed as Log $z = \log r + i\theta + 2n\pi i$, where θ is the principal argument of z and n is an integer.

An equivalent expresion is Log $z = \log r + i\alpha + 2p\pi i$, where α is any argument of z and p is an integer, since $\alpha = 2m\pi + \theta$ for some integer m.

The difference in these two expressions is to be noted.

The first expression gives the p.v. of Log z(*i.e.*, log z) when $n = 0$, but the latter does not give log z when $p = 0$.

Therefore in order to find log z (the p.v. of Log z) we shall necessarily stick to the *polar representation of z with principal argument.*

Note 2. From the definition of Log z it follows that

(i) $\exp(\text{Log } z) = z$ for all $z \neq 0$ and

(ii) one of the values of Log(exp z) is z, the other values being $z + 2n\pi i, n$ being a non-zero integer.

The principal logarithmic function is the *inverse function* of the exponential function and

(i) exp (log z)=z for all $z \neq 0$; and (ii) log (exp z)=z for all $z \in \mathbb{C}$.

Worked Examples.

1. Find Log z and log z, where

(i) $z = 1$, (ii) $z = -1$, (iii) $z = i$, (iv) $z = -i$.

(i) $1 = 1(\cos 0 + i \sin 0)$. (polar form with principal argument)

Log $1 = \log 1 + i(0 + 2n\pi)$, where n is an integer.

Therefore Log $1 = 2n\pi$ and log $1 = 0$. (corresponding to $n = 0$)

(ii) $-1 = 1(\cos \pi + i \sin \pi)$.

Log$(-1) = \log 1 + i(\pi + 2n\pi)$, where n is an integer.

Therefore Log $(-1) = (2n + 1)\pi i$ and log $(-1) = \pi i$. (corresponding to $n = 0$)

(iii) $i = 1(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$.

Log $i = \log 1 + i(\frac{\pi}{2} + 2n\pi)$, where n is an integer.

Therefore Log $i = (4n + 1)\frac{\pi}{2}i$ and log $i = \frac{\pi}{2}i$.

(iv) $-i = 1[\cos \frac{-\pi}{2} + i \sin \frac{-\pi}{2}]$.

Log$(-i) = \log 1 + i(\frac{-\pi}{2} + 2n\pi)$, where n is an integer.

Therefore Log$(-i) = (4n - 1)\frac{\pi}{2}i$ and log$(-i) = \frac{-\pi}{2}i$.

Note that Log$(-i)$ can also be expressed as $(\frac{3\pi}{2} + 2n\pi)i$.

2. Express Log$(x + iy), (x, y) \neq (0, 0)$, in the form $A + iB$ where A and B are real and find log$(x + iy)$.

Since $x + iy$ is a non-zero complex number, it has a polar representation. Let $x + iy = r(\cos \theta + i \sin \theta), -\pi < \theta \leq \pi$.

Then r =mod$(x + iy)$ and $\theta = \arg(x + iy)$ (principal argument).

Let Log$(x + iy) = u + iv$.

Then $\exp(u + iv) = x + iy = r(\cos \theta + i \sin \theta)$

or, $e^u(\cos v + i \sin v) = r(\cos \theta + i \sin \theta)$.

This gives $e^u \cos v = r \cos \theta$, $e^u \sin v = r \sin \theta$.

We have $e^{2u} = r^2 \Rightarrow e^u = r \Rightarrow u = \log r = \frac{1}{2}\log(x^2 + y^2)$.

Since $e^u = r$, we have $\cos v = \cos \theta, \sin v = \sin \theta$.

Therefore $v = \theta + 2n\pi$, where n is an integer.

Hence Log$(x + iy) = u + iv = \frac{1}{2}\log(x^2 + y^2) + i(\theta + 2n\pi)$.

Therefore $A = \frac{1}{2}\log(x^2 + y^2)$ and $B = \theta + 2n\pi$ =Arg$(x + iy)$.

The principal logarithm of $x + iy$ corresponds to $n = 0$ (since θ is the principal argument).

Hence $\log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i \arg(x + iy)$.

3. Find Log z and log z, where $z = 1 + i \tan \theta, \frac{\pi}{2} < \theta < \pi$.

Let $z = r(\cos \phi + i \sin \phi)$ Then $r \cos \phi = 1, r \sin \phi = \tan \theta$.

We have $r^2 = \sec^2 \theta$ and this gives $r = -\sec \theta$ since $\sec \theta < 0$.

Therefore $\cos \phi = -\cos \theta$ and $\sin \phi = -\sin \theta$.

These determine $\phi = \pi + \theta$.

As $\frac{3\pi}{2} < \phi < 2\pi, \phi$ is not the principal argument of z.

$\arg z = \phi - 2\pi = \theta - \pi$.

Hence Log $z = \log(-\sec \theta) + i(\theta - \pi + 2n\pi)$, where n is an integer and log $z = \log(-\sec \theta) + i(\theta - \pi)$.

Properties.

1. If z_1, z_2 be two distinct complex numbers such that $z_1 z_2 \neq 0$, then Log z_1+Log z_2 =Log$(z_1 z_2)$.

Proof. $z_1 z_2 \neq 0 \Rightarrow z_1 \neq 0, z_2 \neq 0$.

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1), z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$.

Then $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$.

Log z_1 = $\log r_1 + i(\theta_1 + 2n\pi)$, where n is an integer ;

Log z_2 = $\log r_2 + i(\theta_2 + 2m\pi)$, where m is an integer ;

Log $(z_1 z_2) = \log(r_1 r_2) + i(\theta_1 + \theta_2 + 2p\pi)$, where p is an integer.

$\text{Log } z_1 + \text{Log } z_2 = \log r_1 + \log r_2 + i(\theta_1 + \theta_2 + 2n\pi + 2m\pi)$
$= \log(r_1 r_2) + i(\theta_1 + \theta_2 + 2q\pi)$, where $q = n + m$.
Since p and q are arbitrary integers, $\text{Log } z_1 + \text{Log } z_2 = \text{Log}(z_1 z_2)$.

Note 1. If $z_1 = z_2$, $\text{Log } z_1 + \text{Log } z_2 = 2 \log r_1 + i(2\theta_1 + 4n\pi)$, where n is an integer; and $\text{Log } z_1 z_2 = 2 \log r_1 + i(2\theta_1 + 2p\pi)$, where p is an integer.

The set of the general values of $\text{Log } z_1 + \text{Log } z_2$ is a proper subset of the set of the general values of $\text{Log } z_1 z_2$.

Hence $\text{Log } z_1 + \text{Log } z_2 \neq \text{Log } z_1 z_2$ if $z_1 = z_2$.

Note 2. $\log z_1 + \log z_2$ is not necessarily equal to $\log(z_1 z_2)$.

For example, let $z_1 = i, z_2 = -1$. Then $z_1 z_2 = -i$.

$|z_1| = 1$, $|z_2| = 1$, $|z_1 z_2| = 1$, $\arg z_1 = \frac{\pi}{2}$, $\arg z_2 = \pi$, $\arg(z_1 z_2) = -\frac{\pi}{2}$.

$\log z_1 = \frac{\pi}{2} i$, $\log z_2 = \pi i$ and $\log(z_1 z_2) = -\frac{\pi}{2} i$.

Hence $\log z_1 + \log z_2 = \frac{3\pi}{2} i \neq \log(z_1 z_2)$.

2. If z_1 and z_2 be two distinct complex numbers such that $z_1 z_2 \neq 0$, then $\text{Log } z_1 - \text{Log } z_2 = \text{Log } \frac{z_1}{z_2}$.

Proof. $z_1 z_2 \neq 0 \Rightarrow z_1 \neq 0, z_2 \neq 0$.

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1), z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$.

Then $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$.

$\text{Log } z_1 = \log r_1 + i(\theta_1 + 2n\pi)$, where n is an integer ;
$\text{Log } z_2 = \log r_2 + i(\theta_2 + 2m\pi)$, where m is an integer ;
$\text{Log } \frac{z_1}{z_2} = \log(\frac{r_1}{r_2}) + i(\theta_1 - \theta_2 + 2p\pi)$, where p is an integer .

$\text{Log } z_1 - \text{Log } z_2 = \log r_1 - \log r_2 + i(\theta_1 - \theta_2 + 2n\pi - 2m\pi)$
$= \log(\frac{r_1}{r_2}) + i(\theta_1 - \theta_2 + 2q\pi)$, where $q = n - m$.

Since p and q are arbitrary integers, $\text{Log } z_1 - \text{Log } z_2 = \text{Log } \frac{z_1}{z_2}$.

Note 1. If $z_1 = z_2$, $\text{Log } z_1 - \text{Log } z_2 = 0$ and $\text{Log } \frac{z_1}{z_2} = \text{Log } 1 = 2n\pi i$, where n is an integer. Hence $\text{Log } z_1 - \text{Log } z_2 \neq \text{Log } \frac{z_1}{z_2}$.

Note 2. $\log z_1 - \log z_2$ is not necessarily equal to $\log \frac{z_1}{z_2}$.

For example, let $z_1 = -1, z_2 = -i$. Then $\frac{z_1}{z_2} = -i$.

$|z_1| = 1$, $|z_2| = 1$, $|\frac{z_1}{z_2}| = 1$, $\arg z_1 = \pi$, $\arg z_2 = -\frac{\pi}{2}$, $\arg(\frac{z_1}{z_2}) = -\frac{\pi}{2}$.

$\log z_1 = \pi i$ $\log z_2 = -\frac{\pi}{2} i$ and $\log(\frac{z_1}{z_2}) = -\frac{\pi}{2} i$.

Hence $\log z_1 - \log z_2 = \frac{3\pi}{2} i \neq \log(\frac{z_1}{z_2})$.

3. If $z \neq 0$ and m be a positive integer, $\text{Log } z^m \neq m \text{Log } z$.

Proof. Let $z = r(\cos \theta + i \sin \theta)$.

Then $z^m = r^m(\cos m\theta + i \sin m\theta)$.

$\text{Log } z = \log r + i(\theta + 2n\pi)$, where n is an integer;
$\text{Log } z^m = \log r^m + i(m\theta + 2p\pi)$, where p is an integer.

$m \text{ Log } z = m \log r + i(m\theta + 2mn\pi)$
$= \log r^m + i(m\theta + 2p_1\pi)$, where $p_1 = mn$.

Since p is arbitrary and p_1 is a multiple of m, each value of $m \text{ Log } z$ is a value of $\text{Log } z^m$ but not conversely.

So the set of values of $m \text{ Log } z$ is a proper subset of the set of values of $\text{Log } z^m$. Therefore $\text{Log } z^m \neq m \text{Log } z$.

For example, let $z = i, m = 2$.
$2 \text{ Log } z = 2 \text{ Log } i = (4n + 1)\pi i$, where n is an integer.
$\text{Log } z^2 = \text{Log}(-1) = (2k + 1)\pi i$, where k is an integer.
Each value of $2 \text{ Log } i$ is a value of $\text{Log } i^2$ but not conversely.
Hence $\text{Log } i^2 \neq 2 \text{ Log } i$.

4. If $z \neq 0$ and m be a positive integer, $\text{Log } z^{1/m} = \frac{1}{m} \text{Log } z$.

Proof. Let $z = r(\cos \theta + i \sin \theta)$.

Then $z^{1/m} = \sqrt[m]{r}(\cos \frac{2k\pi + \theta}{m} + i \sin \frac{2k\pi + \theta}{m})$, where $k = 0, 1, \ldots, m - 1$.

$\text{Log } z = \log r + i(\theta + 2n\pi)$, where p is an integer;

$\text{Log } z^{1/m} = \frac{1}{m} \log r + i(\frac{2k\pi + \theta}{m} + 2p\pi)$, where p is an integer

$= \frac{1}{m} \log r + i[\frac{\theta}{m} + \frac{2(k + mp)}{m}\pi]$.

Since $0 \leq k \leq m - 1$ and p is an arbitrary integer, $k + mp$ is also an arbitrary integer. Let $k + mp = q$.

Then $\text{Log } z^{1/m} = \frac{1}{m} \log r + i(\frac{\theta}{m} + \frac{2q\pi}{m})$, where q is an integer and

$\frac{1}{m} \text{Log} z = \frac{1}{m} \log r + i(\frac{\theta}{m} + \frac{2n\pi}{m})$, where n is an integer.

Therefore $\text{Log } z^{1/m} = \frac{1}{m} \text{Log } z$.

Worked Examples.

1. Verify that $\text{Log}(-i)^{1/2} = \frac{1}{2} \text{Log}(-i)$.

$-i = \cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2})$.

$\frac{1}{2} \text{Log}(-i) = \frac{1}{2}[(2n\pi - \frac{\pi}{2})i] = (n\pi - \frac{\pi}{4})i$, where n is an integer.

Two values of $(-i)^{1/2}$ are $\cos \frac{2k\pi - \frac{\pi}{2}}{2} + i \sin \frac{2k\pi - \frac{\pi}{2}}{2}$, $k = 0, 1$

i.e., $\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}), \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$.

Now $\text{Log } [\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})] = (2m\pi - \frac{\pi}{4})i$, and

$\text{Log } [\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}] = (2p\pi + \frac{3\pi}{4})i = [(2p + 1)\pi - \frac{\pi}{4}]i$, where m, p are integers.

The values of Log $(-i)^{1/2}$ can be exhibited as $(n\pi - \frac{\pi}{4})i$, where n is an integer. Hence Log $(-i)^{1/2} = \frac{1}{2}\text{Log}(-i)$.

2. If x is real prove that $i \log \frac{x-i}{x+i} = \pi - 2\tan^{-1} x$, if $x > 0$
$= -\pi - 2\tan^{-1} x$, if $x \leq 0$.

Case 1. Let $x > 0$.

Let $x + i = r(\cos\theta + i \sin\theta), 0 < \theta < \frac{\pi}{2}$. Then $x = r\cos\theta, 1 = r\sin\theta$. Therefore $\cot\theta = x$.

$\log \frac{x-i}{x+i} = \log \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} = \log[\cos(-2\theta) + i\sin(-2\theta)]$.

$0 < \theta < \frac{\pi}{2} \Rightarrow -\pi < -2\theta < 0 \Rightarrow -2\theta$ is the principal argument.

Therefore $i \log \frac{x-i}{x+i} = i(-2\theta)i = 2\theta$ ... (i)

$\cot\theta = x \Rightarrow \tan(\frac{\pi}{2} - \theta) = x$. $0 < \theta < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - \theta < \frac{\pi}{2}$ and therefore $\tan(\frac{\pi}{2} - \theta) = x$ gives $\frac{\pi}{2} - \theta = \tan^{-1} x$.

From (i) $i \log \frac{x-i}{x+i} = 2\theta = \pi - 2\tan^{-1} x$.

Case 2. Let $x < 0$.

Let $x + i = r(\cos\theta + i \sin\theta), \frac{\pi}{2} < \theta < \pi$. Then $\cot\theta = x$.

$\log \frac{x-i}{x+i} = \log[\cos(-2\theta) + i\sin(-2\theta)]$.

$\frac{\pi}{2} < \theta < \pi \Rightarrow -2\pi < -2\theta < -\pi \Rightarrow 0 < -2\theta + 2\pi < \pi \Rightarrow -2\theta + 2\pi$ is the principal argument.

Therefore $i \log \frac{x-i}{x+i} = i(-2\theta + 2\pi)i = 2\theta - 2\pi$ ... (ii)

$\cot\theta = x \Rightarrow \tan(\frac{\pi}{2} - \theta) = x$. $\frac{\pi}{2} < \theta < \pi \Rightarrow -\pi < -\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < \frac{\pi}{2} - \theta < 0$ and therefore $\tan(\frac{\pi}{2} - \theta) = x$ gives $\frac{\pi}{2} - \theta = \tan^{-1} x$.

From (ii) $i \log \frac{x-i}{x+i} = -2(\pi - \theta) = -\pi - 2\tan^{-1} x$.

Case 3. Let $x = 0$.

$\log \frac{x-i}{x+i} = i\log(-1) = i(\pi i) = -\pi = -\pi - 2\tan^{-1} x$.

3. Prove that $\sin[i \log \frac{a-ib}{a+ib}] = \frac{2ab}{a^2+b^2}$.

Note that $(a, b) \neq (0, 0)$, because otherwise $\frac{a-ib}{a+ib}$ is not defined.

Let $z = \frac{a-ib}{a+ib}$ and let $a + ib = r(\cos\theta + i \sin\theta), -\pi < \theta \leq \pi$.
Then $a = r\cos\theta, b = r\sin\theta$.

Therefore $z = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} = \cos(-2\theta) + i\sin(-2\theta)$.

arg z may not be equal to (-2θ); arg $z = -2\theta + 2k\pi$, where k is an integer such that $-\pi < -2\theta + 2k\pi \leq \pi$.

Therefore $\log z = (-2\theta + 2k\pi)i$, where $-\pi < -2\theta + 2k\pi \leq \pi$.
Hence $\sin(i \log z) = \sin(-2k\pi + 2\theta) = \sin 2\theta = \frac{2ab}{a^2+b^2}$.

4. Express Log [Log $(\cos\theta + i \sin\theta)$]$(0 < \theta < \pi)$ in the form $A + iB$, where A and B are real.

Log $(\cos\theta + i \sin\theta) = (\theta + 2n\pi)i$, where n is an integer.

Case I. Let $n \geq 0$. Then $\theta + 2n\pi > 0$.
mod $[(\theta + 2n\pi)i] = \theta + 2n\pi$ and arg $[(\theta + 2n\pi)i] = \frac{\pi}{2}$.
Therefore Log [Log$(\cos\theta + i \sin\theta)$] =Log $[(\theta + 2n\pi)i]$
$= \log(\theta + 2n\pi) + (2k\pi + \frac{\pi}{2})i$, where k is an integer.

Case II. Let $n < 0$. Then $\theta + 2n\pi < 0$.
mod$[(\theta + 2n\pi)i] = -(\theta + 2n\pi)$ and arg $[(\theta + 2n\pi)i] = -\frac{\pi}{2}$
Therefore Log [Log$(\cos\theta + i \sin\theta)$] =Log$[(\theta + 2n\pi)i]$
$= \log[-(\theta + 2n\pi)] + (2k\pi - \frac{\pi}{2})i$, where k is an integer.

2.14. Complex exponents.

If a be a *non-zero* complex number and z be any complex number, a^z is defined by

$$a^z = \exp(z \text{ Log } a).$$

Since Log a is many-valued, a^z is a many-valued function. The principal value of a^z corresponds to the principal logarithm of a.

Let $a = r(\cos\theta + i \sin\theta), -\pi < \theta \leq \pi$; and $z = x + iy$.
Then z Log $a = (x + iy)[\log r + i(2n\pi + \theta)]$, where n is an integer
$= [x \log r - y(2n\pi + \theta)] + i[x(2n\pi + \theta) + y \log r]$.

Therefore $a^z = e^{x \log r - y(2n\pi+\theta)}[\cos\{x(2n\pi + \theta) + y \log r\} + i \sin\{x(2n\pi + \theta) + y \log r\}]$, where n is an integer.

The principal value of a^z corresponds to $n = 0$.

The p.v. of $a^z = e^{x \log r - y\theta}[\cos(x\theta + y \log r) + i \sin(x\theta + y \log r)]$
$= \exp[(x + iy)(\log r + i\theta)]$
$= \exp(z \log a)$.

Particular cases.

1. Let a be a positive real number and z be a complex number $x + iy$. Then $r = a, \theta = 0$.

$a^z = a^{x+iy} = e^{x \log a - 2n\pi y}[\cos(2n\pi x + y \log a) + i \sin(2n\pi x + y \log a)]$, where n is an integer.

The p.v. of a^z is $e^{x \log a}[\cos(y \log a) + i \sin(y \log a)]$.

In particular, if $a = e$, then

$e^{x+iy} = e^{x-2n\pi y}[\cos(2n\pi x + y) + i \sin(2n\pi x + y)]$, where n is an integer. Therefore e^{x+iy} has many values.

The p.v. of $e^{x+iy} = e^x(\cos y + i \sin y) = \exp(x + iy)$.

Thus e^z is a *many-valued* function and the *principal value* of e^z is $\exp z$.

2. Let a be a negative real number and z be a complex number $x + iy$.

Then $a = r\cos\theta + i \sin\theta$, where $r = b = -a$ and $\theta = \pi$.

$a^{x+iy} = e^{x \log b - (2n+1)\pi y}[\cos\{(2n+1)\pi x + y \log b\} + i \sin\{(2n+1)\pi x + y \log b\}]$, where n is an integer.

The p.v. of $a^{x+iy} = e^{x \log b - \pi y}[\cos(\pi x + y \log b) + i \sin(\pi x + y \log b)]$

$= e^{x \log b}[\cos(y \log b) + i \sin(y \log b)].e^{-\pi y}(\cos \pi x + i \sin \pi x)$.

Now the p.v. of $(-1)^{x+iy} = \exp[(x + iy)\log(-1)]$
$= \exp[(x + iy)\pi i]$
$= e^{-\pi y}(\cos \pi x + i \sin \pi x)$.

Hence the p.v. of a^{x+iy}=(the p.v. of b^{x+iy}).[the p.v. of $(-1)^{x+iy}$].

3. Let a be a positive real number and $z = x$, a real number.

Then $r = a, \theta = 0, y = 0$.

$a^x = e^{x \log a}[\cos(2n\pi x) + i \sin(2n\pi x)]$, where n is an integer and the p.v. of $a^x = e^{x \log a}$.

a^x has infinitely many values having the same modulus $e^{x \log a}$. In the complex plane they are represented by points on a circle whose centre is the origin and radius is $e^{x \log a}$.

Subcase (i). Let x be an integer.

Then $\cos(2n\pi x) + i \sin(2n\pi x) = 1$ and a^x is the unique real number $e^{x \log a}$.

Subcase (ii). Let x be a rational number $\frac{p}{q}$, where p and q are integers prime to each other $(q > 1)$.

Then $\cos(\frac{2n\pi p}{q}) + i \sin(\frac{2n\pi p}{q})$ has only q distinct values and they correspond to $n = 0, 1, \ldots, q-1$.

Therefore $a^{\frac{p}{q}}$ has only a finite number of values.

The principal value of $a^{\frac{p}{q}}$ is $e^{\frac{p}{q} \log a}$ (corresponding to $n = 0$).

Subcase (iii). Let x be an irrational real number.

Then $\sin(2n\pi x) \neq 0$ for all integers $n \neq 0$. Therefore a^x has infinite number of values, all of which excepting the principal value are non-real complex numbers. The principal value of a^x is $e^{x \log a}$.

4. Let a be a negative real number and $z = x$, a real number.

Let $a = -b$, where $b > 0$. Then $r = b, \theta = \pi, y = 0$.

$a^x = e^{x \log b}[\cos(2n+1)\pi x + i \sin(2n+1)\pi x]$, where n is an integer.

The p.v. of a^x is $e^{x \log b}[\cos \pi x + i \sin \pi x]$.

Now the p.v. of $(-1)^x = \exp[x(\pi i)]$
$= (\cos \pi x + i \sin \pi x)$.

Hence the p.v. of a^x =(the p.v. of b^x).[the p.v. of $(-1)^x$].

5. Let $a = \cos\theta + i \sin\theta$, where θ is real and $z = x + iy$.

Then $a^z = (\cos\theta + i \sin\theta)^{x+iy}$
$= \exp[(x + iy)\ \mathrm{Log}\ (\cos\theta + i \sin\theta)]$
$= \exp[(x + iy)(2n\pi + \theta)i]$
$= \exp[-y(2n\pi + \theta) + x(2n\pi + \theta)i]$
$= e^{-y(2n\pi+\theta)}[\cos x(2n\pi + \theta) + i \sin x(2n\pi + \theta)]$
$= e^{-y\ \mathrm{Arg}\ a}[\cos(x\ \mathrm{Arg}\ a) + i \sin(x\ \mathrm{Arg}\ a)]$.

Hence the p.v. of $(\cos\theta + \sin\theta)^{x+iy}$
$= e^{-y \arg a}[\cos(x \arg a) + i \sin(x \arg a)]$.

5a. Let $a = \cos\theta + i \sin\theta$, where θ is real and $z = x$, a real number.

Then $a^x = (\cos\theta + i \sin\theta)^x = [\cos x(2n\pi + \theta) + i \sin x(2n\pi + \theta)]$, where n is an integer
and the p.v. of $(\cos\theta + i \sin\theta)^x = \cos(x \arg a) + i \sin(x \arg a)$, where $a = \cos\theta + i \sin\theta$.

Subcase (i). Let x be an integer.

Then $(\cos\theta + i \sin\theta)^x = \cos x\theta + i \sin x\theta$. In this case $(\cos\theta + i \sin\theta)^x$ has only one value.

Subcase (ii). Let x be a rational number $\frac{p}{q}$, where p and q are integers prime to each other $(q > 1)$.

Then $\cos(2n\pi + \theta)\frac{p}{q} + i \sin(2n\pi + \theta)\frac{p}{q}$ has only q distinct values.

So in this case $(\cos\theta + i \sin\theta)^x$ has only a finite number of distinct values and $\cos x\theta + i \sin x\theta$ (corresponding to $n = 0$) is one of these.

The principal value of $(\cos\theta + i \sin\theta)^{\frac{p}{q}}$ is $\cos(\frac{p}{q} \arg a) + i \sin(\frac{p}{q} \arg a)$, where $a = \cos\theta + i \sin\theta$.

Subcase (iii). Let x be an irrational real number, then $(\cos\theta + i \sin\theta)^x = \cos x(2n\pi + \theta) + i \sin x(2n\pi + \theta)$, where n is an integer and $\cos x\theta + i \sin x\theta$ (corresponding to $n = 0$) is one of these values.

The principal value of $(\cos\theta + i \sin\theta)^x$ is $\cos x \arg a + i \sin x \arg a$, where $a = \cos\theta + i \sin\theta$.

Thus if $a = \cos\theta + i\sin\theta$, where $-\pi < \theta \leq \pi$,

(i) the principal value of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ is $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$, where p, q are positive integers prime to each other;

(ii) the principal value of $(\cos\theta + i\sin\theta)^x$ is $\cos x\theta + i\sin x\theta$, if x be irrational.

Examples.

1. if x, y be real,

$1^{x+iy} = \exp[(x+iy)\text{Log } 1] = \exp[(x+iy)(2n\pi i)]$, where n is an integer
$= e^{-2n\pi y}[\cos 2n\pi x + i\sin 2n\pi x]$.

In particular, if $y = 0$, then

$1^x = [\cos 2n\pi x + i\sin 2n\pi x]$, where n is an integer.

(i) If x be an integer, $[\cos 2n\pi x + i\sin 2n\pi x] = 1$ for all n. Therefore $1^x = 1$.

(ii) If x be a rational number $\frac{p}{q}$, where p, q are integers prime to each other $(q > 1)$, $[\cos 2n\pi.\frac{p}{q} + i\sin 2n\pi.\frac{p}{q}]$ has q distinct values. Therefore there are q distinct values of 1^x in this case.

(iii) If x be an irrational number, $1^x = [\cos 2n\pi x + i\sin 2n\pi x]$, where n is an integer.

Since x is irrational, $\sin 2n\pi x \neq 0$ for all integers $n \neq 0$. For $n = 0$, $\sin 2n\pi x = 0$ and $\cos 2n\pi x = 1$. Therefore there are infinite number of values of 1^x of which one is real and all others are non-real.

2. If x, y be real,

$(-1)^{x+iy} = \exp[(x+iy)\text{Log } (-1)]$
$= \exp[(x+iy)(2n+1)\pi i)]$, where n is an integer
$= e^{-(2n+1)\pi y}[\cos(2n+1)\pi x + i\sin(2n+1)\pi x]$.

In particular, if $y = 0$, then

$(-1)^x = [\cos(2n+1)\pi x + i\sin(2n+1)\pi x]$, where n is an integer.

(i) If x be an integer, $[\cos(2n+1)\pi x + i\sin(2n+1)\pi x] = \pm 1$ according as x is even or odd. Therefore $(-1)^x = \pm 1$ according as x is even or odd.

(ii) If x be a rational number $\frac{p}{q}$, where p, q are integers prime to each other $(q > 1)$, $[\cos(2n+1)\pi.\frac{p}{q} + i\sin(2n+1)\pi.\frac{p}{q}]$ has q distinct values. Therefore there are q distinct values of $(-1)^x$ in this case.

(iii) If x be an irrational number, $(-1)^x = [\cos(2n+1)\pi x + i\sin(2n+1)\pi x]$, where n is an integer.

Since x is irrational, $\sin(2n+1)\pi x \neq 0$ for all integers n. Therefore there are infinite number of values of $(-1)^x$ all of which are non-real.

Properties.

1. If z_1, z_2 and a are complex numbers where $a \neq 0$,

$$a^{z_1}.a^{z_2} \neq a^{z_1+z_2},$$

but (the p.v. of a^{z_1}).(the p.v. of a^{z_2}) = the p.v. of $a^{z_1+z_2}$.

Proof. $a^{z_1} = \exp(z_1\ \text{Log}\ a)$
$= \exp[z_1(\log a + 2n\pi i)]$, where n is an integer;

$a^{z_2} = \exp[z_2(\log a + 2m\pi i)]$, where m is an integer;

$a^{z_1+z_2} = \exp[(z_1+z_2)(\log a + 2p\pi i)]$
$= \exp[(z_1+z_2)\log a + 2p(z_1+z_2)\pi i]$, where p is an integer.

$$\begin{aligned} a^{z_1}.a^{z_2} &= \exp[z_1(\log a + 2n\pi i)].\exp[z_2(\log a + 2m\pi i)] \\ &= \exp[z_1(\log a + 2n\pi i) + z_2(\log a + 2m\pi i)] \\ &= \exp[(z_1+z_2)\log a + 2(nz_1 + mz_2)\pi i]. \end{aligned}$$

When m, n, p are arbitrary integers, the set of complex numbers $p(z_1+z_2)$ is a subset of the set of complex numbers $nz_1 + mz_2$, but not conversely.

Therefore $a^{z_1}.a^{z_2} \neq a^{z_1+z_2}$.

But the p.v. of $a^{z_1} = \exp(z_1\ \log a)$, the p.v. of $a^{z_2} = \exp(z_2\ \log a)$ and the p.v. of $a^{z_1+z_2} = \exp[(z_1+z_2)\log a]$.

Therefore (the p.v. of a^{z_1}).(the p.v. of a^{z_2}) = the p.v. of $a^{z_1+z_2}$.

2. If z, a and b are complex numbers and $ab \neq 0$, $(ab)^z = a^z.b^z$, but the p.v. of $(ab)^z \neq$ (the p.v. of a^z).(the p.v. of b^z).

Proof. $a^z = \exp(z\ \text{Log}\ a)$, $b^z = \exp(z\ \text{Log}\ b)$,

$(ab)^z = \exp[z\ \text{Log}(ab)]$.

$$\begin{aligned} \exp[z\ \text{Log}\ (ab)] &= \exp(z\ \text{Log}\ a + z\ \text{Log}\ b) \\ &= \exp(z\ \text{Log}\ a).\exp(z\ \text{Log}\ b). \\ \text{Therefore } (ab)^z &= a^z.b^z. \end{aligned}$$

The p.v. of $a^z = \exp(z\ \log a)$, the p.v. of $b^z = \exp(z\ \log b)$ and the p.v. of $(ab)^z = \exp(z\ \log ab)$.

But $\log ab \neq \log a + \log a$, in general.

Therefore the p.v. of $(ab)^z \neq$ (the p.v. of a^z).(the p.v. of b^z).

3. If a and z are complex numbers and $a \neq 0$,

$\text{Log}\ a^z = z\ \text{Log}\ a + 2n\pi i$, where n is an integer.

Proof. Let $z = x + iy, a = r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leq \pi$.

Then $z\ \text{Log}\ a = (x+iy)[\log r + i(\theta + 2m\pi)]$, where m is an integer
$= x\log r - y(2m\pi + \theta) + i[x(2m\pi+\theta) + y\ \log r]$;

and $a^z = e^{x\ \log r - y(2m\pi+\theta)}[\cos\{x(2m\pi+\theta) + y\ \log r\} + i\ \sin\{x(2m\pi+\theta) + y\ \log r\}]$.

Hence $\text{Log } a^z = [x \log r - y(2m\pi+\theta)] + i[x(2m\pi+\theta) + y \log r + 2n\pi]$, where n is an integer

$= z \text{ Log } a + 2n\pi i.$

Worked Examples.

1. Find the general values of i^i.

$i^i = \exp(i \text{ Log } i)$
$= \exp[i(2n\pi + \frac{\pi}{2})i]$, where n is an integer
$= \exp[-(4n+1)\frac{\pi}{2}] = e^{-(4n+1)\frac{\pi}{2}}$.

Note. The values of i^i are all real.

2. Find the principal value of $(1+i)^i$

$1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4})$.

$\text{Log } (1+i) = \frac{1}{2}\log 2 + i(2n\pi + \frac{\pi}{4})$, where n is an integer and $\log(1+i) = \frac{1}{2}\log 2 + \frac{\pi}{4}i$.

Hence the p.v. of $(1+i)^i = \exp[i \log(1+i)]$
$= \exp[-\frac{\pi}{4} + \frac{1}{2}(\log 2)i]$
$= e^{-\frac{\pi}{4}}[\cos(\frac{1}{2}\log 2) + i \sin(\frac{1}{2}\log 2)]$.

3. Show that the ratio of the principal values of $(x+iy)^{a+ib}$ and $(x-iy)^{a-ib}$, where x, y, a, b are real, is

(i) $\cos 2(b \log r + a\theta) + i \sin 2(b \log r + a\theta)$, where $r = |x+iy|$, $\theta = \arg(x+iy) \neq \pi$;

(ii) $e^{-2b\pi}[\cos(2b \log r) + i \sin(2b \log r)]$, where $r = |x|$, $\arg(x+iy) = \pi$.

Let $x + iy = r(\cos\theta + i \sin\theta), -\pi < \theta \leq \pi$. Then $r = |x+iy|$, $\theta = \arg(x+iy)$. $x - iy = r(\cos\theta - i\sin\theta) = r[\cos(-\theta) + i\sin(-\theta)]$.
$\arg(x-iy) = -\theta$ provided $\theta \neq \pi$. If $\theta = \pi$, then $\arg(x-iy) = \pi$.

Case 1. $\theta = \arg(x+iy) \neq \pi$.

The p.v. of $(x+iy)^{a+ib} = \exp[(a+ib)\log(x+iy)]$
$= \exp[(a+ib)(\log r + i\theta)]$
$= \exp[(a \log r - b\theta) + i(b \log r + a\theta)]$.

The p.v. of $(x-iy)^{a-ib} = \exp[(a-ib)\log(x-iy)]$
$= \exp[(a-ib)(\log r - i\theta)]$
$= \exp[(a \log r - b\theta) - i(b \log r + a\theta)]$.

The ratio $= \exp[2i(b \log r + a\theta)]$
$= \cos 2(b \log r + a\theta) + i \sin 2(b \log r + a\theta)$.

Case 2. $\theta = \arg(x+iy) = \pi$.

The p.v.of $(x+iy)^{a+ib} = \exp[(a \log r - b\pi) + i(b \log r + a\pi)]$.

The p.v. of $(x-iy)^{a-ib} = \exp[(a-ib)(\log r + i\pi)]$
$= \exp[(a \log r + b\pi) - i(b \log r - a\pi)]$.

The ratio $= \exp[-2b\pi + 2ib \log r]$
$= e^{-2b\pi}[\cos(2b \log r) + i \sin(2b \log r)]$.

4. Discuss the reality of x^y, where x, y are both irrational.

Case 1. $x > 0$.

$x^y = \exp[y \text{ Log } x] = \exp[y(\log x + 2n\pi i)]$, where n is an integer
$= e^{y \log x}[\cos 2ny\pi + \sin 2ny\pi]$.

For $n = 0$, $\sin 2ny\pi = 0$ and $\cos 2ny\pi = 1$. $\sin 2ny\pi \neq 0$ for all $n \neq 0$.

Therefore x^y has infinite number of values of which only one is real and all others are non-real.

Case 2. $x < 0$. Let $x = -t$, where $t > 0$.

$x^y = (-t)^y = \exp[y \text{ Log } (-t)] = \exp[y\{\log t + (2n+1)\pi i\}]$
$= e^{y \log t}[\cos(2n+1)y\pi + i \sin(2n+1)y\pi]$.

Since y is irrational, $\sin(2n+1)y\pi$ is non-zero for all integers n.

Therefore x^y has infinite number of values all of which are non-real.

2.14.1. Definition of $\text{Log}_a z$, where a and z are non-zero complex numbers.

We define $\text{Log}_a z = w$ such that z is *any* value of a^w.

We have $a^w = \exp(w \text{ Log } a)$. Therefore $z = \exp(w \text{ Log } a)$.

This gives $\text{Log } z = w \text{ Log } a$

or, $w = \dfrac{\text{Log } z}{\text{Log } a}$.

Thus w is a doubly infinitely many-valued function of z.

For example,

(i) $\text{Log}_i(-1) = \frac{(2n+1)\pi i}{(4m+1)\frac{\pi}{2}i}$, where m and n are integers;

(ii) $\text{Log}_e(1) = \frac{2n\pi i}{1+2m\pi i}$, where m and n are integers;

(iii) $\text{Log}_e(-1) = \frac{(2n+1)\pi i}{1+2m\pi i}$, where m and n are integers.

Exercises 2B

1. (i) If $\exp z$ is positive real, prove that
$Im\ z = 2n\pi$, where n is an integer.

(ii) If $\exp z$ is negative real, prove that
$Im\ z = (2n+1)\pi$, where n is an integer.

(iii) If $\exp z$ is purely imaginary, prove that
$Im\ z = (2n+1)\frac{\pi}{2}$, where n is an integer.

2. Find all values of z such that
(i) $\exp z = -2$, (ii) $\exp z = 4i$, (iii) $\exp \bar{z} = 1 + i$,
(iv) $\exp(2z + \bar{z}) = 3 + 4i$.

3. If $a\exp(i\theta) + b\exp(3i\theta) = c$ where a, b, c are all real, prove that either $a + b = \pm c$, or $b(b - a) = c^2$.

4. Prove that
(i) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$,
(ii) $\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} + \cos\frac{7\pi}{13} + \cos\frac{9\pi}{13} + \cos\frac{11\pi}{13} = \frac{1}{2}$.

5. Find (i) Log 4, log 4; (ii) Log (-4), log (-4);
(iii) Log $4i$, log $4i$; (iv) Log $(-4i)$, log $(-4i)$.

6. Show that (i) $\log i + \log(1 + i) = \log i(1 + i)$,
(ii) $\log i + \log(-1 + i) \neq \log i(-1 + i)$,
(iii) $\text{Log}(1 + i)^3 \neq 3\ \text{Log}\ (1 + i)$ but $\log(1 + i)^3 = 3\log(1 + i)$.

7. Express in the form $A + iB$ where A and B are real
(i) $\log(\sin\theta + i\cos\theta)$, $0 < \theta < \frac{\pi}{2}$
(ii) $\log(\sin\theta - i\cos\theta)$, $0 < \theta < \frac{\pi}{2}$
(iii) $\log(1 + i\cot\theta)$, $\frac{\pi}{2} < \theta < \pi$
(iv) $\log(1 + \cos 2\theta + i\sin 2\theta)$, $\frac{\pi}{2} < \theta < \pi$.

8. If $a > 0, b > 0$ and $z = \frac{(a-b)+i(a+b)}{(a+b)+i(a-b)}$, show that

$$\log z = i\tan^{-1}\frac{2ab}{a^2-b^2} \quad \text{if } a > b$$
$$= i(\pi + \tan^{-1}\frac{2ab}{a^2-b^2}) \quad \text{if } a < b$$
$$= i\frac{\pi}{2} \quad \text{if } a = b.$$

[**Hint.** Let $z = r(\cos\theta + i\sin\theta)$, $-\pi < \theta \leq \pi$. Then $r = 1, \cos\theta = \frac{a^2-b^2}{a^2+b^2}, \sin\theta = \frac{2ab}{a^2+b^2}$. If $a > b, \theta = \tan^{-1}\frac{2ab}{a^2-b^2}$. If $a < b, \theta = \pi + \tan^{-1}\frac{2ab}{a^2-b^2}$.]

9. If x be real, prove that $i\log\frac{x+i}{x-i} = -\pi + 2\tan^{-1}x$, if $x \geq 0$.
$= \pi + 2\tan^{-1}x$, if $x < 0$.

10. If x be real, prove that $i\log\frac{1+ix}{1-ix} = -2\tan^{-1}x$.

11. Show that
(i) if z be a non-zero complex number, $\text{Log}\frac{1}{z} = -\text{Log}\ z$ but $\log\frac{1}{z}$ may not be equal to $-\log z$;
(ii) if a and z are complex numbers and $a \neq 0$, $a^{-z} = 1/a^z$ and also the p.v. of a^{-z} = the p.v. of $1/a^z$.

12. Find the general values and the principal value of
(i) 2^2, (ii) $2^{\frac{1}{2}}$, (iii) $2^{\sqrt{2}}$, (iv) $(-1)^{\sqrt{2}}$,
(v) 3^i, (vi) $(-1)^i$, (vii) $(1+i)^{1+i}$, (viii) $i^{\log(1+i)}$.

13. Show that
(i) $\text{Log}_i i = \frac{4n+1}{4m+1}$, where m, n are integers.
(ii) $\text{Log}_2(-1) = \frac{(2n+1)\pi i}{\log 2 + 2m\pi i}$, where m, n are integers.

14. Show that
(i) the p.v. of $(\frac{i}{-i})^i$ is equal to the ratio of the principal values of i^i and $(-i)^i$;
(ii) the p.v. of $(\frac{-i}{i})^i$ is not equal to the ratio of the principal values of $(-i)^i$ and i^i.

15. Show that (i) $\sqrt{i(-1+i)} \neq \sqrt{i}.\sqrt{-1+i}$; (ii) $\sqrt{i(1-i)} = \sqrt{i}.\sqrt{1-i}$.

16. Show that the ratio of the principal values of $(1+i)^{1-i}$ and $(1-i)^{1+i}$ is $\sin(\log 2) + i\cos(\log 2)$.

17. If a, b, x are real and $|a + ib| = 1$, prove that $(a + ib)^{ix}$ is purely real.

18. If x be a non-zero real number, prove that
$x^i = e^{-2n\pi}[\cos(\log x) + i\sin(\log x)]$, when $x > 0$
$= e^{-(2n+1)\pi}[\cos\log(-x) + i\sin\log(-x)]$, when $x < 0$.

19. If a, b are positive real numbers and $a^z = b$, show that the general values of z are given by $z = \frac{\log b + 2m\pi i}{\log a + 2n\pi i}$, m, n being integers.

20. If $i^z = i$ show that z is real and the general values of z are given by
$z = \frac{4m+1}{4n+1}$, m, n being integers.

21. Find the general values and the principal value of i^{x+iy} where x, y are real.

Show that the principal value is purely real or purely imaginary according as x is an even or an odd integer.

22. Find the general values of $[\sigma(\cos\psi + i\sin\psi)]^{ix}$ where $\sigma > 0, -\pi < \psi \leq \pi$ and x is a non-zero real number. Show that the points representing them in the complex plane are collinear.

23. (i) Show that $(\sqrt{2})^{\sqrt{3}}$ has infintely many values, all of which excepting the principal value, are non-real complex numbers.

Show that the points representing them lie on a circle in the complex plane.

(ii) Show that $(-\sqrt{2})^{\sqrt{3}}$ has infintely many values, all of which are non-real complex numbers and the points representing them lie on a circle in the complex plane.

24. Find the general values of $[\sigma(\cos\psi + i\sin\psi)^{u+iv}]$, where $\sigma > 0, -\pi < \psi \leq \pi$, u and v are real.

If $u \neq 0, v \neq 0$ show that the points represented by them lie on the equiangular spiral $r = \sigma^{\frac{(u^2+v^2)}{u}} . e^{\frac{-v}{u}\theta}$ in the complex plane.

[**Hint.** Let $[\sigma(\cos\psi + i\sin\psi)]^{u+iv} = \rho(\cos\phi + i\sin\phi)$.
Then $\rho = e^{u\log\sigma - v(2n\pi+\psi)}, \phi = v\log\sigma + u(2n\pi+\psi)$.
Therefore $\rho = e^{u\log\sigma - (\phi - v\log\sigma)v/u} = \sigma^{\frac{u^2+v^2}{u}} . e^{\frac{-v}{u}\phi}$.]

2.15. Trigonometric functions.

In Art. 2.12 we have seen that when y is real,

$\exp(iy) = \cos y + i\sin y$.

Therefore $\exp(-iy) = \cos y - i\sin y$.

These relations determine $\cos y$ and $\sin y$ in terms of the exponential function.

When x is real, $\cos x = \frac{\exp(ix)+\exp(-ix)}{2}$ $\quad \sin x = \frac{\exp(ix)-\exp(-ix)}{2i}$

When z is complex, cosine and sine functions are defined by

$$\cos z = \frac{\exp(iz)+\exp(-iz)}{2}, \quad \sin z = \frac{\exp(iz)-\exp(-iz)}{2i}.$$

The other trigonometric functions are defined by

$$\tan z = \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z = \frac{1}{\sin z}, \quad \cot z = \frac{\cos z}{\sin z}.$$

Properties.

1. When z is a complex number, $\cos^2 z + \sin^2 z = 1$.

Proof. $\cos z = \frac{t+\frac{1}{t}}{2}$ and $\sin z = \frac{t-\frac{1}{t}}{2i}$, where $t = \exp(iz)$.

Therefore $\cos^2 z + \sin^2 z = \frac{1}{4}\{(t+\frac{1}{t})^2 + (t-\frac{1}{t})^2\} = 1$.

2. If z_1, z_2 be complex numbers then

$\sin(z_1+z_2) = \sin z_1\cos z_2 + \cos z_1\sin z_2$,

$\cos(z_1+z_2) = \cos z_1\cos z_2 - \sin z_1\sin z_2$.

Proof. $\sin(z_1+z_2) = \frac{\exp[i(z_1+z_2)] - \exp[-i(z_1+z_2)]}{2i}$

$$= \frac{\exp(iz_1).\exp(iz_2) - \exp(-iz_1).\exp(-iz_2)}{2i}$$

$$= \frac{t_1t_2 - \frac{1}{t_1t_2}}{2i}, \text{ where } t_1 = \exp(iz_1), t_2 = \exp(iz_2)$$

$$= \frac{t_1^2t_2^2 - 1}{2i\,t_1t_2}$$

$$= \frac{(t_1^2-1)(t_2^2+1) + (t_1^2+1)(t_2^2-1)}{4i\,t_1t_2}$$

$$= \frac{(t_1 - \frac{1}{t_1})}{2i}.\frac{(t_2+\frac{1}{t_2})}{2} + \frac{(t_1+\frac{1}{t_1})}{2}.\frac{(t_2-\frac{1}{t_2})}{2i}$$

$$= \sin z_1\cos z_2 + \cos z_1\sin z_2.$$

$\cos(z_1+z_2) = \frac{\exp[i(z_1+z_2)] + \exp[-i(z_1+z_2)]}{2}$

$$= \frac{\exp(iz_1).\exp(iz_2) + \exp(-iz_1).\exp(-iz_2)}{2}$$

$$= \frac{t_1t_2 + \frac{1}{t_1t_2}}{2}, \text{ where } t_1 = \exp(iz_1), t_2 = \exp(iz_2)$$

$$= \frac{t_1^2t_2^2 + 1}{2\,t_1t_2}$$

$$= \frac{(t_1^2+1)(t_2^2+1) + (t_1^2-1)(t_2^2-1)}{4\,t_1t_2}$$

$$= \frac{(t_1+\frac{1}{t_1})}{2}.\frac{(t_2+\frac{1}{t_2})}{2} - \frac{(t_1-\frac{1}{t_1})}{2i}.\frac{(t_2-\frac{1}{t_2})}{2i}$$

$$= \cos z_1\cos z_2 - \sin z_1\sin z_2.$$

3. $\sin(z+\pi) = -\sin z$, $\cos(z+\pi) = -\cos z$, $\tan(z+\pi) = \tan z$,

$\sin(z+2\pi) = \sin z$, $\cos(z+2\pi) = \cos z$.

Proof. $\sin(z+\pi) = \sin z\cos\pi + \cos z\sin\pi = -\sin z$

$\cos(z+\pi) = \cos z\cos\pi - \sin z\sin\pi = -\cos z$

$\tan(z+\pi) = \frac{\sin(z+\pi)}{\cos(z+\pi)} = \frac{-\sin z}{-\cos z} = \tan z$

$\sin(z+2\pi) = \sin z\cos 2\pi + \cos z\sin 2\pi = \sin z$

$\cos(z+2\pi) = \cos z\cos 2\pi - \sin z\sin 2\pi = \cos z$.

Note. $\sin z$ and $\cos z$ are periodic functions of period 2π; $\tan z$ is a periodic function of period π.

4. If x, y are real,

$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$,

$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$.

Proof. $\sin(x+iy) = \frac{\exp[i(x+iy)] - \exp[-i(x+iy)]}{2i}$

$= \frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)}{2i}$

$= \sin x \cdot \frac{e^{y}+e^{-y}}{2} - \cos x \cdot \frac{e^{y}-e^{-y}}{2i}$

$= \sin x \cosh y + i \cos x \sinh y$.

$\cos(x+iy) = \frac{\exp[i(x+iy)] + \exp[-i(x+iy)]}{2}$

$= \frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)}{2}$

$= \cos x \cdot \frac{e^{y}+e^{-y}}{2} - i \sin x \cdot \frac{e^{y}-e^{-y}}{2}$

$= \cos x \cosh y - i \sin x \sinh y$.

The right hand side expressions give the real and imaginary parts of $\sin z$ and $\cos z$, when z is a complex number.

It follows that

$$\begin{aligned} |\sin(x+iy)|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y \\ &= \sin^2 x + \sinh^2 y. \end{aligned}$$

$$\begin{aligned} |\cos(x+iy)|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x \cosh^2 y + (1 - \cos^2 x) \sinh^2 y \\ &= \cos^2 x + \sinh^2 y. \end{aligned}$$

Since $\sinh y$ increaes steadily with y, it follows that the functions $\sin z$ and $\cos z$ are not bounded in absolute value. But if x be real, the functions $\sin x$ and $\cos x$ are bounded in absolute value, as $|\sin x|$ and $|\cos x|$ are never greater than 1.

5. When z is a complex number,

$\sin \bar{z} = \overline{\sin z}$, $\cos \bar{z} = \overline{\cos z}$ and $\tan \bar{z} = \overline{\tan z}$.

Proof. Let $z = x + iy$, where x, y are real.

$$\begin{aligned} \sin z = \sin(x+iy) &= \sin x \cosh y + i \cos x \sinh y. \\ \sin \bar{z} = \sin(x-iy) &= \sin x \cosh(-y) + i \cos x \sinh(-y) \\ &= \sin x \cosh y - i \cos x \sinh y \\ &= \overline{\sin z}. \end{aligned}$$

Similarly, $\cos \bar{z} = \overline{\cos z}$ and $\tan \bar{z} = \frac{\sin \bar{z}}{\cos \bar{z}} = \frac{\overline{\sin z}}{\overline{\cos z}} = \overline{\left(\frac{\sin z}{\cos z}\right)} = \overline{\tan z}$.

2.16. Hyperbolic functions.

When x is real, the hyperbolic functions $\cosh x$, $\sinh x$, ... are defined by

$\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$, $\tanh x = \frac{\sinh x}{\cosh x}$

$\operatorname{sech} x = \frac{1}{\cosh x}$, $\operatorname{cosech} x = \frac{1}{\sinh x}$, $\coth x = \frac{\cosh x}{\sinh x}$.

Properties.

1. $\cosh^2 x - \sinh^2 x = 1$, $\operatorname{sech}^2 x + \tanh^2 x = 1$, $\coth^2 x - \operatorname{cosech}^2 x = 1$.

Proof follows from the definition.

2. $\cosh(-x) = \cosh x$, $\sinh(-x) = -\sinh x$, $\tanh(-x) = -\tanh x$.

Proof follows from the definition.

3. $\cosh x \geq 1$ for all real x.

Proof. Considering two positive numbers e^x and e^{-x} and applying A.M. $\geq$ G.M., we have

$\frac{e^x + e^{-x}}{2} \geq \sqrt{e^x . e^{-x}}$, the equality occurs when $e^x = e^{-x}$.

or, $\cosh x \geq 1$, the equality occurs when $x = 0$.

4. For all real x, $\sinh x$ increases steadily with x and assumes every real value only once.

Proof. $\frac{d}{dx}(\sinh x) = \frac{e^x + e^{-x}}{2} = \cosh x$.

Since $\frac{d}{dx}(\sinh x) > 0$ for all real x, $\sinh x$ increases steadily with x.

Let t be an arbitrary real number.

$\sinh x = t \Rightarrow e^x - e^{-x} = 2t$

or, $e^{2x} - 2t\, e^x - 1 = 0$.

Therefore $e^x = t + \sqrt{t^2+1}$, since $e^x > 0$

or, $x = \log(t + \sqrt{t^2+1})$.

Therefore for an arbitrary real t, x is a unique real number.

When z is complex, the hyperbolic functions $\cosh z$, $\sinh z$, ... are defined by

$\cosh z = \frac{\exp z + \exp(-z)}{2}$, $\sinh z = \frac{\exp z - \exp(-z)}{2}$, $\tanh z = \frac{\sinh z}{\cosh z}$

$\operatorname{sech} z = \frac{1}{\cosh z}$, $\operatorname{cosech} z = \frac{1}{\sinh z}$, $\coth z = \frac{\cosh z}{\sinh z}$.

Properties (continued).

5. When z is a complex number, $\cosh^2 z - \sinh^2 z = 1$.

Proof. $\cosh z = \frac{t+\frac{1}{t}}{2}$ and $\sinh z = \frac{t-\frac{1}{t}}{2}$, where $t = \exp z$.

Hence $\cosh^2 z - \sinh^2 z = \frac{1}{4}\{(t+\frac{1}{t})^2 - (t+\frac{1}{t})^2\} = \frac{1}{4}.4 = 1$.

6. If z_1, z_2 be complex numbers,

$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$,

$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$.

Proof. $\sinh(z_1 + z_2) = \frac{\exp(z_1+z_2) - \exp(-z_1-z_2)}{2}$

$= \frac{\exp z_1 . \exp z_2 - \exp(-z_1).\exp(-z_2)}{2}$

$= \frac{t_1 t_2 - \frac{1}{t_1 t_2}}{2}$, where $t_1 = \exp z_1, t_2 = \exp z_2$

$= \frac{t_1^2 t_2^2 - 1}{2 t_1 t_2}$

$= \frac{(t_1^2-1)(t_2^2+1)+(t_1^2+1)(t_2^2-1)}{4 t_1 t_2}$

$= \frac{(t_1 - \frac{1}{t_1})}{2} \cdot \frac{(t_2+\frac{1}{t_2})}{2} + \frac{(t_1+\frac{1}{t_1})}{2} \cdot \frac{(t_2-\frac{1}{t_2})}{2}$

$= \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$.

Proof of the second part left to the reader.

7. When z is a complex number,

$\cos(iz) = \cosh z$, $\sin(iz) = i \sinh z$,

$\cosh(iz) = \cos z$, $\sinh(iz) = i \sin z$.

These follow from definitions.

8. If x, y are real numbers,

(i) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$,

(ii) $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$.

Proof. $\sinh(x + iy) = \frac{\exp(x+iy) - \exp(-x-iy)}{2}$

$= \frac{e^x(\cos y + i \sin y) - e^{-x}(\cos y - i \sin y)}{2}$

$= \frac{e^x - e^{-x}}{2} \cos y + i \frac{e^x + e^{-x}}{2} \sin y$

$= \sinh x \cos y + i \cosh x \sin y$.

(ii) Proof left to the reader.

9. $\cosh z$ **and** $\sinh z$ are periodic functions of period $2\pi i$, $\tanh z$ is a periodic function of period πi.

Proof. $\cosh(z + 2k\pi i) = \cosh z \cosh(2k\pi i) + \sinh z \sinh(2k\pi i)$

$= \cosh z \cos(2k\pi) + i \sinh z \sin(2k\pi)$

$= \cosh z$, if k be an integer.

$\sinh(z + 2k\pi i) = \sinh z \cosh(2k\pi i) + \cosh z \sinh(2k\pi i)$

$= \sinh z \cos(2k\pi) + i \cosh z \sin(2k\pi)$

$= \sinh z$, if k be an integer.

It follows that $\cosh z$ and $\sinh z$ are periodic functions of period $2\pi i$.

$\cosh(z + k\pi i) = \cosh z \cos(k\pi) + i \sinh z \sin(k\pi)$

$= (-1)^k \cosh z$, if k be an integer.

$\sinh(z + k\pi i) = \sinh z \cos(k\pi) + i \cosh z \sin(k\pi)$

$= (-1)^k \sinh z$, if k be an integer.

Therefore $\tanh(z + k\pi i) = \tanh z$, for all integers k and so $\tanh z$ is a periodic function of period πi.

Worked Examples.

1. Find all values of z such that $\cos z = 0$.

Let $z = x + iy$.

Then $\cos z = 0 \Rightarrow \cos x \cosh y = 0$ ... .. (i)

and $\sin x \sinh y = 0$ ... ... (ii)

From (i) $\cos x = 0$, since $\cosh y \neq 0$.

Therefore $x = (2n+1)\frac{\pi}{2}$, where n is an integer.

From (ii) $\sinh y = 0$, since $\sin x = \sin(2n+1)\frac{\pi}{2} \neq 0$.

Therefore $y = 0$.

Hence $z = (2n+1)\frac{\pi}{2}$, where n is an integer.

2. Find all values of z such that $\sin z = 0$.

Let $z = x + iy$.

Then $\sin z = 0 \Rightarrow \sin x \cosh y = 0$ ... ... (i)

and $\cos x \sinh y = 0$ ... ... (ii)

From (i) $\sin x = 0$, since $\cosh y \neq 0$.

Therefore $x = n\pi$, where n is an integer.

From (ii) $\sinh y = 0$, since $\cos x = \cos n\pi \neq 0$.

Therefore $y = 0$. Hence $z = n\pi$, where n is an integer.

3. Find the general solution of $\cos z = 2$.

$\cos z = 2$
$\Rightarrow t + \frac{1}{t} = 4$, where $t = \exp(iz)$
$\Rightarrow t^2 - 4t + 1 = 0$.

Therefore $t = 2 \pm \sqrt{3}$.

When $t = 2+\sqrt{3}$, $iz = \text{Log}\,(2+\sqrt{3}) = \log(2+\sqrt{3}) + 2n\pi i$.

Therefore $z = 2n\pi - i\log(2+\sqrt{3})$.

When $t = 2-\sqrt{3}$, $iz = \text{Log}\,(2-\sqrt{3}) = \log(2-\sqrt{3}) + 2n\pi i$.

Therefore $z = 2n\pi - i\log(2-\sqrt{3}) = 2n\pi + i\log(2+\sqrt{3})$.

Combining, $z = 2n\pi \pm i\log(2+\sqrt{3})$, n being an integer.

4. Find the general solution of $\sin z = 2$.

$\sin z = 2$
$\Rightarrow t - \frac{1}{t} = 4i$, where $t = \exp(iz)$
$\Rightarrow t^2 - 4it - 1 = 0$.

Therefore $t = (2 \pm \sqrt{3})i$.

When $t = (2+\sqrt{3})i$, $iz = \text{Log}\,(2+\sqrt{3})i = \log(2+\sqrt{3}) + (2n\pi + \frac{\pi}{2})i$.

Therefore $z = 2n\pi + \{\frac{\pi}{2} - i\log(2+\sqrt{3})\}$.

When $t = (2-\sqrt{3})i$, $iz = \text{Log}\,(2-\sqrt{3})i = \log(2-\sqrt{3}) + (2n\pi + \frac{\pi}{2})i$.

Therefore $z = 2n\pi + \frac{\pi}{2} - i\log(2-\sqrt{3}) = (2n+1)\pi - \{\frac{\pi}{2} - i\log(2+\sqrt{3})\}$.

Combining, $z = n\pi + (-1)^n\{\frac{\pi}{2} - i\log(2+\sqrt{3})\}$, n being an integer.

5. Find the general solution of $\tan z = 2 + i$.

$\tan z = (2+i)$

$\Rightarrow \frac{t-\frac{1}{t}}{t+\frac{1}{t}} = i(2+i)$, where $t = \exp(iz)$
$\Rightarrow t^2 = \frac{2i}{2-2i} = -\frac{1}{2} + \frac{1}{2}i$.

Therefore $2iz = \text{Log}(-\frac{1}{2} + \frac{1}{2}i) = \frac{1}{2}\log\frac{1}{2} + (2n\pi + \frac{3\pi}{4})i$, since $|-\frac{1}{2} + \frac{1}{2}i| = \frac{1}{\sqrt{2}}$ and the principal amplitude of $(-\frac{1}{2} + \frac{1}{2}i)$ is $\frac{3\pi}{4}$.

Therefore $z = n\pi + \frac{3\pi}{8} + \frac{i}{4}\log 2$, n being an integer.

6. Find the general solution of $\cosh z = -2$.

$\cosh z = -2$
$\Rightarrow t + \frac{1}{t} = -4$, where $t = \exp z$
$\Rightarrow t^2 + 4t + 1 = 0$.

Therefore $t = -2 \pm \sqrt{3}$.

When $t = -2+\sqrt{3}$, $z = \text{Log}\,(-2+\sqrt{3}) = \log(2-\sqrt{3}) + (2n\pi + \pi)i$.

When $t = -2-\sqrt{3}$, $z = \text{Log}\,(-2-\sqrt{3}) = \log(2+\sqrt{3}) + (2n\pi + \pi)i$.

Combining, we have $z = \log(2 \pm \sqrt{3}) + (2n+1)\pi i$.

7. Find the general solution of $\sinh z = 2i$.

$\sinh z = 2i$
$\Rightarrow t - \frac{1}{t} = 4i$, where $t = \exp z$
$\Rightarrow t^2 - 4it - 1 = 0$.

Therefore $t = (2 \pm \sqrt{3})i$.

When $t = (2+\sqrt{3})i$, $z = \text{Log}\,(2+\sqrt{3})i = \log(2+\sqrt{3}) + (2n\pi + \frac{\pi}{2})i$.

When $t = (2-\sqrt{3})i$, $z = \text{Log}\,(2-\sqrt{3})i = \log(2-\sqrt{3}) + (2n\pi + \frac{\pi}{2})i$.

Combining, we have $z = \log(2 \pm \sqrt{3}) + (2n\pi + \frac{\pi}{2})i$.

8. If $\tan(\theta + i\phi) = \tan\beta + i\sec\beta$ where θ, ϕ, β are real and $0 < \beta < \pi$, show that $e^{2\phi} = \cot\frac{\beta}{2}$ and $\theta = n\pi + \frac{\pi}{4} + \frac{\beta}{2}$.

$$\frac{\sin(\theta+i\phi)}{\cos(\theta+i\phi)} = \frac{\sin\beta + i}{\cos\beta}$$

or, $$\frac{\cos(\theta+i\phi)+i\sin(\theta+i\phi)}{\cos(\theta+i\phi)-i\sin(\theta+i\phi)} = \frac{\cos\beta+i\sin\beta-1}{\cos\beta-i\sin\beta+1}$$

or, $$\frac{\exp i(\theta+i\phi)}{\exp -i(\theta+i\phi)} = \frac{-2\sin^2\frac{\beta}{2} + 2i\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{2\cos^2\frac{\beta}{2} - 2i\sin\frac{\beta}{2}\cos\frac{\beta}{2}}$$

or, $$\exp 2i(\theta + i\phi) = \frac{2i\sin\frac{\beta}{2}(\cos\frac{\beta}{2} + i\sin\frac{\beta}{2})}{2\cos\frac{\beta}{2}(\cos\frac{\beta}{2} - i\sin\frac{\beta}{2})}$$

or, $e^{-2\phi}(\cos 2\theta + i\sin 2\theta) = i\tan\frac{\beta}{2}(\cos\beta + i\sin\beta)$.

Therefore $e^{-2\phi}\cos 2\theta = -\tan\frac{\beta}{2}\sin\beta$, $e^{-2\phi}\sin 2\theta = \tan\frac{\beta}{2}\cos\beta$.

We have $e^{-4\phi} = \tan^2\frac{\beta}{2}$ and this implies $e^{-2\phi} = \tan\frac{\beta}{2}$, since $\tan\frac{\beta}{2} > 0$, i.e., $e^{2\phi} = \cot\frac{\beta}{2}$.

Also $\cos 2\theta = -\sin\beta$ and $\sin 2\theta = \cos\beta$.

These determine $2\theta = 2n\pi + \frac{\pi}{2} + \beta$, i.e., $\theta = n\pi + \frac{\pi}{4} + \frac{\beta}{2}$.

9. If $\log \sin(\theta + i\phi) = \alpha + i\beta$ where $\theta, \phi, \alpha, \beta$ are real, prove that

(i) $2\cos 2\theta = e^{2\phi} + e^{-2\phi} - 4e^{2\alpha}$

(ii) $\cos(\theta - \beta) = e^{2\phi}\cos(\theta + \beta)$.

Let $\sin(\theta + i\phi) = \rho(\cos\psi + i\sin\psi)$, where $-\pi < \psi \leq \pi$.
Then $\log \sin(\theta + i\phi) = \log\rho + i\psi$ and therefore $\alpha = \log\rho, \beta = \psi$.

We have $\rho\cos\psi = \sin\theta\cosh\phi$, $\rho\sin\psi = \cos\theta\sinh\phi$ and therefore $\rho^2 = \sin^2\theta\cosh^2\phi + \cos^2\theta\sinh^2\phi = \sin^2\theta + \sinh^2\phi$.

$2\alpha = 2\log\rho = \log(\sin^2\theta + \sinh^2\phi)$.

Therefore $e^{2\alpha} = \sin^2\theta + \sinh^2\phi = \frac{1}{2}(1 - \cos 2\theta) + (\frac{e^{\phi} - e^{-\phi}}{2})^2$

or, $4e^{2\alpha} = 2 - 2\cos 2\theta + e^{2\phi} + e^{-2\phi} - 2$

or, $2\cos 2\theta = e^{2\phi} + e^{-2\phi} - 4e^{2\alpha}$.

Again, $\cos\beta = \frac{\sin\theta\cosh\phi}{\rho}$, $\sin\beta = \frac{\cos\theta\sinh\phi}{\rho}$.

Therefore $\tan\beta\tan\theta = \tanh\phi$

or, $\frac{1+\tan\beta\tan\theta}{1-\tan\beta\tan\theta} = \frac{1+\tanh\phi}{1-\tanh\phi}$

or, $\frac{\cos(\theta-\beta)}{\cos(\theta+\beta)} = \frac{\cosh\phi+\sinh\phi}{\cosh\phi-\sinh\phi}$

or, $\cos(\theta - \beta) = e^{2\phi}\cos(\theta + \beta)$.

10. If $x = \log\tan(\frac{\pi}{4} + \frac{\theta}{2})$, where θ is real, prove that
$\theta = -i\,\mathrm{Log}\tan(\frac{\pi}{4} + i\frac{x}{2})$.

Since θ is real, x is real.

$e^x = \tan(\frac{\pi}{4} + \frac{\theta}{2}) = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$.

Therefore $\tan\frac{\theta}{2} = \frac{e^x-1}{e^x+1}$

or, $\frac{t-t^{-1}}{t+t^{-1}} = i.\frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}}$, where $t = \exp(\frac{i\theta}{2})$

or, $\frac{t^2-1}{t^2+1} = \frac{i\sinh\frac{x}{2}}{\cosh\frac{x}{2}} = \frac{\sin\frac{ix}{2}}{\cos\frac{ix}{2}} = \tan\frac{ix}{2}$

or, $t^2 = \frac{1+\tan\frac{ix}{2}}{1-\tan\frac{ix}{2}}$

or, $\exp(i\theta) = \tan(\frac{\pi}{4} + \frac{ix}{2})$

or, $\theta = -i\,\mathrm{Log}\tan(\frac{\pi}{4} + i\frac{x}{2})$.

11. If $z = x + iy$, prove that

(i) $|\sinh y| \leq |\sin z| \leq \cosh y$;

(ii) $|\sinh y| \leq |\cos z| \leq \cosh y$.

(i) $\sin z = \sin(x + iy) = \sin x\cosh y + i\cos x\sinh y$.

Therefore $|\sin z|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$
$= \sin^2 x\cosh^2 y + (1 - \sin^2 x)\sinh^2 y$
$= \sin^2 x + \sinh^2 y$
$\leq 1 + \sinh^2 y = \cosh^2 y$.

Therefore $|\sin z| \leq \cosh y$, since $\cosh y$ is positive for all real y.

$|\sin z|^2 = \sin^2 x + \sinh^2 y \Rightarrow |\sin z|^2 \geq \sinh^2 y$

or, $|\sin z| \geq |\sinh y|$.

Therefore $|\sinh y| \leq |\sin z| \leq \cosh y$.

(ii) Similar proof.

2.17. Inverse functions.

1. Let z be a given complex number and w be a complex number such that $\sin w = z$. Then w is said to be *an inverse sine* of z.

$\sin w = z \Rightarrow \cos w = \pm\sqrt{1 - z^2}$.

Then $\exp(iw) = iz \pm \sqrt{1 - z^2}$

or, $w = -i\,\mathrm{Log}(iz \pm \sqrt{1 - z^2})$.

Since Log z is a multiple-valued function, w is a multiple-valued function. The values of w are denoted by $\mathrm{Sin}^{-1}z$.

The *principal value* of w is obtained by choosing $\cos w = \sqrt{1 - z^2}$ (the *principal square root* of $1 - z^2$) and taking the *principal logarithm*. It is denoted by $\sin^{-1}z$.

Therefore $\mathrm{Sin}^{-1}z = -i\,\mathrm{Log}(iz \pm \sqrt{1 - z^2})$

and $\sin^{-1}z = -i\log(iz + \sqrt{1 - z^2})$.

2. Let z be a given complex number and w be a complex number such that $\cos w = z$. Then w is said to be *an inverse cosine* of z.

$\cos w = z \Rightarrow \sin w = \pm\sqrt{1 - z^2}$.

Then $\exp(iw) = z \pm i\sqrt{1 - z^2}$

or, $w = -i\,\mathrm{Log}(z \pm i\sqrt{1 - z^2})$.

Since Log z is a multiple-valued function of z, w is a multiple-valued function. The values of w are denoted by $\mathrm{Cos}^{-1}z$.

The *principal value* of w is obtained by choosing $\sin w = \sqrt{1-z^2}$ (the *principal square root* of $1-z^2$) and taking the *principal logarithm*. It is denoted by $\cos^{-1} z$.

Therefore $\text{Cos}^{-1} z = -i \text{ Log}(z \pm i\sqrt{1-z^2})$

and $\cos^{-1} z = -i \log(z + i\sqrt{1-z^2})$.

3. Let z be a given complex number and w be a complex number such that $\tan w = z$. Then w is said to be *an inverse tangent* of z.

$\tan w = z \Rightarrow \frac{t-t^{-1}}{t+t^{-1}} = iz$, where $t = \exp(iw)$

or, $(1-iz)t^2 - (1+iz) = 0$.

t cannot be determined if $1 - iz = 0$, i.e., if $z = -i$.

Again, since $t = \exp(iw) \neq 0$, therefore $1 + iz \neq 0$, i.e., $z \neq i$.

Therefore if $z \neq \pm i$, $t^2 = \frac{1+iz}{1-iz}$

or, $\exp(2iw) = \frac{1+iz}{1-iz}$

or, $w = -\frac{1}{2}i \text{ Log} \frac{1+iz}{1-iz}$.

w is a multiple-valued function. The values of w are denoted by $\text{Tan}^{-1} z$.

The *principal value* of w is obtained by taking the *principal logarithm*. It is denoted by $\tan^{-1} z$.

Therefore $\text{Tan}^{-1} z = -\frac{1}{2}i \text{ Log } \frac{1+iz}{1-iz}$, provided $z \neq \pm i$

and $\tan^{-1} z = -\frac{1}{2}i \log \frac{1+iz}{1-iz}$, provided $z \neq \pm i$.

In a similar manner the inverse of other trigonometric functions can be defined.

4. Let z be a given complex number and w be a complex number such that $\sinh w = z$. Then w is said to be *an inverse sinh* of z.

$\sinh w = z \Rightarrow t - \frac{1}{t} = 2z$ where $t = \exp w$

$\Rightarrow t = z \pm \sqrt{z^2+1}$.

Therefore $w = \text{Log}(z \pm \sqrt{z^2+1})$.

w is a multiple-valued function. The values of w are denoted by $\text{Sinh}^{-1} z$.

Therefore $\text{Sinh}^{-1} z = \text{Log}(z \pm \sqrt{z^2+1})$.

5. Let z be a given complex number and w be a complex number such that $\cosh w = z$. Then w is said to be an *inverse cosh* of z.

$\cosh w = z \Rightarrow t + \frac{1}{t} = 2z$ where $t = \exp w$

$\Rightarrow t = z \pm \sqrt{z^2-1}$.

Therefore $w = \text{Log}(z \pm \sqrt{z^2-1})$.

w is a multiple-valued function. The values of w are denoted by $\text{Cosh}^{-1} z$.

Therefore $\text{Cosh}^{-1} z = \text{Log}(z \pm \sqrt{z^2-1})$.

6. Let z be a given complex number and w be a complex number such that $\tanh w = z$. Then w is said to be *an inverse tanh* of z.

$\tanh w = z \Rightarrow \frac{t-t^{-1}}{t+t^{-1}} = z$, where $t = \exp(w)$

or, $(1-z)t - (1+z)\frac{1}{t} = 0$.

or, $(1-z)t^2 - (1+z) = 0$.

t cannot be determined if $z = 1$.

Again, since $t = \exp(w) \neq 0$, therefore $1 + z \neq 0$, i.e., $z \neq -1$.

Therefore if $z \neq \pm 1$, , $t^2 = \frac{1+z}{1-z}$ or, $w = \frac{1}{2} \text{ Log} \frac{1+z}{1-z}$.

w is a multiple-valued function. The values of w are denoted by $\text{Tanh}^{-1} z$.

Therefore $\text{Tanh}^{-1} z = \frac{1}{2} \text{ Log} \frac{1+z}{1-z}$, provided $z \neq \pm 1$.

In a similar manner $\text{Cosech}^{-1} z$, $\text{Sech}^{-1} z$, $\text{Coth}^{-1} z$ can be defined.

Worked Examples.

1. Find $\text{Cos}^{-1}(2)$, $\cos^{-1}(2)$.

Let $\text{Cos}^{-1}(2) = z$. Then $\cos z = 2$.

$\sin^2 z + \cos^2 z = 1$ gives $\sin z = \pm\sqrt{3}i$.

We have $\exp(iz) = \cos z + i \sin z = 2 \pm \sqrt{3}$.

When $\exp(iz) = 2 - \sqrt{3}$, $z = -i \text{ Log } (2-\sqrt{3})$

$= -i[\log(2-\sqrt{3}) + 2n\pi i]$

$= 2n\pi - i \log(2-\sqrt{3})$

$= 2n\pi + i \log(2+\sqrt{3})$.

When $\exp(iz) = 2 + \sqrt{3}$, $z = -i \text{ Log } (2+\sqrt{3})$

$= -i[\log(2+\sqrt{3}) + 2n\pi i]$

$= 2n\pi - i \log(2+\sqrt{3})$.

Therefore $\text{Cos}^{-1}(2) = 2n\pi \pm i \log(2+\sqrt{3})$,
and $\cos^{-1}(2) = i \log(2+\sqrt{3})$.

2. Find $\text{Sin}^{-1}(2)$, $\sin^{-1}(2)$.

Let $\text{Sin}^{-1}(2) = z$. Then $\sin z = 2$.

$\sin^2 z + \cos^2 z = 1$ gives $\cos z = \pm\sqrt{3}i$

We have $\exp(iz) = \cos z + i \sin z = (2 \pm \sqrt{3})i$.

When $\exp(iz) = (2+\sqrt{3})i, z = -i \text{ Log } (2+\sqrt{3})i$
$= -i[\log(2+\sqrt{3}) + (2n\pi + \frac{\pi}{2})i]$
$= (2n\pi + \frac{\pi}{2}) - i \log(2+\sqrt{3})$
$= 2n\pi + \{\frac{\pi}{2} - i \log(2+\sqrt{3})\}$.

When $\exp(iz) = (2-\sqrt{3})i, z = -i \text{ Log } (2-\sqrt{3})i$
$= -i[\log(2-\sqrt{3}) + (2n\pi + \frac{\pi}{2})i]$
$= (2n\pi + \frac{\pi}{2}) - i \log(2-\sqrt{3})$
$= (2n+1)\pi - \{\frac{\pi}{2} - i \log(2+\sqrt{3})\}$

Therefore $\text{Sin}^{-1}(2) = n\pi + (-1)^n\{\frac{\pi}{2} - i \log(2+\sqrt{3})\}$,
and $\sin^{-1}(2) = \frac{\pi}{2} - i \log(2+\sqrt{3})$.

3. Find $\text{Cos}^{-1}(-1)$, $\cos^{-1}(-1)$.

Let $\text{Cos}^{-1}(-1) = z$. Then $\cos z = -1$.

or, $\frac{t+\frac{1}{t}}{2} = -1$, where $t = \exp iz$

or, $t^2 + 2t + 1 = 0$

or, $t = -1, -1$

or, $\exp iz = -1$.

Therefore $iz = \text{Log}(-1)$
$= \log 1 + (\pi + 2n\pi)i$, where n is an integer.
or, $z = \pi + 2n\pi$, where n is an integer.

Therefore $\text{Cos}^{-1}(-1) = \pi + 2n\pi$, where n is an integer
and $\cos^{-1}(-1) = \pi$.

4. Find $\text{Cos}^{-1}(i)$, $\cos^{-1}(i)$.

Let $\text{Cos}^{-1}(i) = z$. Then $\cos z = i$.

$\sin^2 z + \cos^2 z = 1$ gives $\sin z = \pm\sqrt{2}$.

We have $\exp(iz) = \cos z + i \sin z = (1 \pm \sqrt{2})i$.

When $\exp(iz) = (1+\sqrt{2})i, z = -i \text{ Log } (\sqrt{2}+1)i$
$= -i[\log(\sqrt{2}+1) + (2n\pi + \frac{\pi}{2})i]$
$= 2n\pi + \frac{\pi}{2} - i \log(\sqrt{2}+1)$.

When $\exp(iz) = (1-\sqrt{2})i, z = -i \text{ Log } (1-\sqrt{2})i$
$= -i[\log(\sqrt{2}-1) + (2n\pi - \frac{\pi}{2})i]$
$= (2n\pi - \frac{\pi}{2}) - i \log(\sqrt{2}-1)$
$= 2n\pi - \{\frac{\pi}{2} - i \log(\sqrt{2}+1)\}$.

Therefore $\text{Cos}^{-1}(i) = 2n\pi \pm \{\frac{\pi}{2} - i \log(\sqrt{2}+1)\}$,
and $\cos^{-1}(i) = \frac{\pi}{2} - i \log(\sqrt{2}+1)$.

5. Find $\text{Sin}^{-1}(i)$, $\sin^{-1}(i)$.

Let $\text{Sin}^{-1}(i) = z$. Then $\sin z = i$.

$\sin^2 z + \cos^2 z = 1$ gives $\cos z = \pm\sqrt{2}$.

We have $\exp(iz) = \cos z + i \sin z = -1 \pm \sqrt{2}$.

When $\exp(iz) = \sqrt{2}-1, z = -i \text{ Log } (\sqrt{2}-1)$
$= -i[\log(\sqrt{2}-1) + 2n\pi i]$
$= 2n\pi - i \log(\sqrt{2}-1)$
$= 2n\pi + i \log(\sqrt{2}+1)$.

When $\exp(iz) = -\sqrt{2}-1, z = -i \text{ Log } (-\sqrt{2}-1)$
$= -i[\log(\sqrt{2}+1) + (2n\pi + \pi)i]$
$= (2n+1)\pi - i \log(\sqrt{2}+1)$.

Therefore $\text{Sin}^{-1}(i) = n\pi + (-1)^n \log(\sqrt{2}+1)i$
and $\sin^{-1}(i) = i \log(\sqrt{2}+1)$.

6. Find $\text{Tan}^{-1}(-1)$ and $\tan^{-1}(-1)$.

$\tan z = -1 \Rightarrow \frac{t-\frac{1}{t}}{t+\frac{1}{t}} = -i$, where $t = \exp(iz)$.

Therefore $t^2 = \frac{1-i}{1+i} = -i$ or, $\exp(2iz) = -i$.

This gives $2iz = \text{Log}(-i) = (2n\pi - \frac{1}{2}\pi)i$
or, $z = n\frac{\pi}{2} - \frac{\pi}{4}$.

Therefore $\text{Tan}^{-1}(-1) = n\frac{\pi}{2} - \frac{\pi}{4}$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$.

7. Find z such that $\tan z = \cos\theta + i \sin\theta, 0 \le \theta < \frac{\pi}{2}$.

$\tan z = \cos\theta + i \sin\theta$

$\Rightarrow \frac{t-\frac{1}{t}}{t+\frac{1}{t}} = i(\cos\theta + i \sin\theta)$, where $t = \exp(iz)$

or, $t^2 = \frac{1+i\cos\theta-\sin\theta}{1-i\cos\theta+\sin\theta} = \frac{i\cos\theta}{1+\sin\theta}$.

Therefore $\exp(2iz) = \frac{\cos\theta}{1+\sin\theta}i$

or, $2iz = \text{Log}(\frac{\cos\theta}{1+\sin\theta}i)$

$= \log\frac{\cos\theta}{1+\sin\theta} + (2n\pi + \frac{\pi}{2})i$, since $\frac{\cos\theta}{1+\sin\theta} > 0$.

Therefore $z = n\pi + \frac{\pi}{4} - \frac{i}{2}\log\frac{\cos\theta}{1+\sin\theta}$

$= n\pi + \frac{\pi}{4} + \frac{i}{2}\log\frac{1+\sin\theta}{\cos\theta}$

$= n\pi + \frac{\pi}{4} + \frac{i}{2}\log(\sec\theta + \tan\theta)$.

8. If $\cosh^{-1}(x+iy) + \cosh^{-1}(x-iy) = \cosh^{-1}a$, where x, y, a are real and $a > 1$, prove that the point (x, y) lies on an ellipse.

Let $\cosh^{-1}(x+iy) = u + iv$. Then $\cosh(u+iv) = x + iy$.

Therefore $\cosh u \cos v = x$, $\sinh u \sin v = y$.

Then $\cosh(u - iv) = x - iy$

or, $\cosh^{-1}(x-iy) = u - iv + 2k\pi i$, where k is some integer.

Therefore $(u+iv) + (u - iv + 2k\pi i) = \cosh^{-1}a$

or, $\cosh(2u + 2k\pi i) = a$

or, $\cosh 2u = a$.

Now $\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1$

or, $\frac{x^2}{\frac{a+1}{2}} + \frac{y^2}{\frac{a-1}{2}} = 1$.

Since $a > 1$, the point (x, y) lies on an ellipse.

9. If $\tan^{-1}(x+iy) = \alpha + i\beta$ where x, y, α, β are real and $(x, y) \neq (0, \pm 1)$, prove that

(i) $x^2 + y^2 + 2x\cot 2\alpha = 1$;

(ii) $x^2 + y^2 + 1 - 2y\coth 2\beta = 0$.

If $(x, y) = (0, \pm 1)$, $\tan^{-1}(x+iy)$ is not defined.

$\tan^{-1}(x+iy) = \alpha + i\beta$ implies $\tan(\alpha + i\beta) = x + iy$.

Therefore $\tan(\alpha - i\beta) = x - iy$, since $\tan\bar{z} = \overline{\tan z}$.

$\cot 2\alpha = \frac{1}{\tan[(\alpha+i\beta)+(\alpha-i\beta)]}$

$= \frac{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}$

$= \frac{1-x^2-y^2}{2x}$.

Therefore $2x\cot 2\alpha = 1 - x^2 - y^2$

or, $x^2 + y^2 + 2x\cot 2\alpha = 1$.

Again $\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$

$= \frac{2iy}{1+x^2+y^2}$.

But $\tan(2i\beta) = \frac{\sin(2i\beta)}{\cos(2i\beta)} = \frac{i\sinh 2\beta}{\cosh 2\beta} = i\tanh 2\beta$.

Therefore $(1 + x^2 + y^2)\tanh 2\beta = 2y$

or, $1 + x^2 + y^2 - 2y\coth 2\beta = 0$.

10. Express $\tan^{-1}(x+iy)$ in the form $A + iB$, where A, B, x, y are real and $(x, y) \neq (0, \pm 1)$.

$\tan^{-1}(x+iy) = -\frac{1}{2}i\log\frac{1+i(x+iy)}{1-i(x+iy)}$

$= -\frac{1}{2}i\log\frac{1-y+ix}{1+y-ix}$

$= -\frac{1}{2}i\log\frac{(1-y+ix)(1+y+ix)}{1+x^2+y^2+2y}$

$= -\frac{1}{2}i\log\frac{(1-x^2-y^2)+2ix}{1+x^2+y^2+2y}$.

Let $\frac{(1-x^2-y^2)+2ix}{1+x^2+y^2+2y} = r(\cos\theta + i\sin\theta)$, $-\pi < \theta \leq \pi$.

Then $r^2 = \frac{(1-x^2-y^2)^2+4x^2}{(1+x^2+y^2+2y)^2} = \frac{(1+x^2+y^2)^2-4y^2}{(1+x^2+y^2+2y)^2} = \frac{1+x^2+y^2-2y}{1+x^2+y^2+2y}$

and $\tan\theta = \frac{2x}{1-x^2-y^2}$.

Therefore $\theta = \tan^{-1}\frac{2x}{1-x^2-y^2}$ if $1 - x^2 - y^2 > 0$

$= \pi + \tan^{-1}\frac{2x}{1-x^2-y^2}$ if $1 - x^2 - y^2 < 0, x \geq 0$

$= -\pi + \tan^{-1}\frac{2x}{1-x^2-y^2}$ if $1 - x^2 - y^2 < 0, x < 0$

$= \frac{\pi}{2}$ if $x^2 + y^2 = 1, x > 0$

$= -\frac{\pi}{2}$ if $x^2 + y^2 = 1, x < 0$... ... ... (i)

Hence $\tan^{-1}(x+iy) = -\frac{1}{2}i[\frac{1}{2}\log\frac{1+x^2+y^2-2y}{1+x^2+y^2+2y} + i\theta]$

$= \frac{1}{2}\theta + \frac{i}{4}\log\frac{1+x^2+y^2+2y}{1+x^2+y^2-2y}$, θ being given by (i).

Some paticular examples.

(i) $\tan^{-1}(1+i) = -\frac{1}{2}i\log\frac{1+i(1+i)}{1-i(1+i)} = -\frac{1}{2}i\log\frac{-1+2i}{5}$.

Let $\frac{-1+2i}{5} = r(\cos\theta + i\sin\theta)$, $-\pi < \theta \leq \pi$.

Then $r = \frac{1}{\sqrt{5}}, \theta = \pi + \tan^{-1}(-2)$.

$\tan^{-1}(1+i) = -\frac{1}{2}i[\log\frac{1}{\sqrt{5}} + i(\pi + \tan^{-1}(-2))] = \frac{1}{2}\pi + \frac{1}{2}\tan^{-1}(-2) - \frac{1}{4}i\log 5$.

(ii) $\tan^{-1}(1+\sqrt{2}i) = -\frac{1}{2}i\log\frac{1+i(1+\sqrt{2}i)}{1-i(1+\sqrt{2}i)} = -\frac{1}{2}i\log\frac{-1+i}{2+\sqrt{2}}$.

Let $\frac{-1+i}{2+\sqrt{2}} = r(\cos\theta + i\sin\theta)$, $-\pi < \theta \le \pi$. Then $r = \frac{1}{\sqrt{2}+1}, \theta = \frac{3\pi}{4}$.

$\tan^{-1}(1+\sqrt{2}i) = -\frac{1}{2}i[\log\frac{1}{\sqrt{2}+1} + \frac{3\pi}{4}i] = \frac{3\pi}{8} + \frac{1}{2}i\log(\sqrt{2}+1)$.

(iii) $\tan^{-1}(\cos\alpha + i\sin\alpha)$, where $0 \le \alpha < \frac{\pi}{2}$.

Comparing with $\tan^{-1}(x+iy)$, here $x^2+y^2 = 1, x > 0$. Therefore $\theta = \frac{\pi}{2}$.

Here $\frac{1+x^2+y^2+2y}{1+x^2+y^2-2y} = \frac{1+y}{1-y} = \frac{1+\sin\alpha}{1-\sin\alpha} = (\sec\alpha + \tan\alpha)^2$.

Therefore $\tan^{-1}(\cos\alpha + i\sin\alpha) = \frac{\pi}{4} + \frac{i}{4}\log(\sec\alpha + \tan\alpha)^2 = \frac{\pi}{4} + \frac{i}{2}\log(\sec\alpha + \tan\alpha)$.

(iv) $\tan^{-1}(\cos\alpha + i\sin\alpha)$, where $\frac{\pi}{2} < \alpha \le \pi$.

Comparing with $\tan^{-1}(x+iy)$, here $x^2+y^2 = 1, x < 0$. Therefore $\theta = -\frac{\pi}{2}$.

Here $\frac{1+x^2+y^2+2y}{1+x^2+y^2-2y} = \frac{1+y}{1-y} = \frac{1+\sin\alpha}{1-\sin\alpha} = (\sec\alpha + \tan\alpha)^2$.

Therefore $\tan^{-1}(\cos\alpha + i\sin\alpha) = -\frac{\pi}{4} + \frac{i}{4}\log(\sec\alpha + \tan\alpha)^2)$.

[Note that here $(\sec\alpha + \tan\alpha) < 0$.]

Another method.

$$\tan^{-1}(\cos\alpha + i\sin\alpha) = \frac{1}{2i}\log\frac{1+i(\cos\alpha+i\sin\alpha)}{1-i(\cos\alpha+i\sin\alpha)}$$
$$= \frac{1}{2i}\log\frac{1-\sin\alpha+i\cos\alpha}{1+i\sin\alpha-i\cos\alpha}$$
$$= \frac{1}{2i}\log\frac{(1-\sin\alpha+i\cos\alpha)(1+\sin\alpha+i\cos\alpha)}{(1+\sin\alpha)^2+\cos^2\alpha}$$
$$= \frac{1}{2i}\log\frac{i\cos\alpha}{1+i\sin\alpha}$$
$$= \frac{1}{2i}[\log\frac{-\cos\alpha}{1+i\sin\alpha} - \frac{\pi}{2}i], \text{ since } \frac{\cos\alpha}{1+\sin\alpha} > 0$$
$$= -\frac{\pi}{4} + \frac{1}{2}i\log(-\sec\alpha - \tan\alpha).$$

2.18. Gregory's series.

When θ is real, $\exp(i\theta) = \cos\theta + i\sin\theta$
and $\exp(-i\theta) = \cos\theta - i\sin\theta$.

Therefore $\exp(2i\theta) = \frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta} = \frac{1+i\tan\theta}{1-i\tan\theta}$ provided $\theta \ne (2n+1)\frac{\pi}{2}$.

Therefore $\text{Log}\frac{1+i\tan\theta}{1-i\tan\theta} = 2i\theta$

or, $\text{Log}(1+i\tan\theta) - \text{Log}(1-i\tan\theta) = 2i\theta$

or, $\log(1+i\tan\theta) - \log(1-i\tan\theta) + 2n\pi i = 2i\theta$.

For a complex number z, the expansion of $\log(1+z)$ as the infinite series $z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ is valid when

(i) $|z| < 1$, and (ii) $|z| = 1$ but amp $z \ne \pi$.

Therefore $\log(1+i\tan\theta) = i\tan\theta - \frac{i^2\tan^2\theta}{2} + \frac{i^3\tan^3\theta}{3} - \cdots$

and $\log(1-i\tan\theta) = -i\tan\theta - \frac{i^2\tan^2\theta}{2} - \frac{i^3\tan^3\theta}{3} - \cdots$

provided $-1 \le \tan\theta \le 1$.

Therefore $2i\theta = 2n\pi i + 2i[\tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots]$

or, $\theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots$

provided $-1 \le \tan\theta \le 1$.

Taking principal value, $\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots$

provided $-1 \le \tan\theta \le 1$ and $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

Let $\tan\theta = x$.

Therefore $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \cdots$

if $-1 \le x \le 1$ and $-\frac{\pi}{4} \le \tan^{-1} x \le \frac{\pi}{4}$.

Note. When z is complex

$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$ provided $|z| \le 1$ but $z \ne \pm i$,

and also the principal value of $\tan^{-1} z$ is taken into account.

Worked Examples.

1. Prove that $\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots \cdots$

We have $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \cdots$ when $-1 \le x \le 1$.

Therefore $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \cdots$

The series in the right hand side is convergent. Therefore grouping of terms is valid.

Therefore $\frac{\pi}{4} = (1 - \frac{1}{3}) + (\frac{1}{5} - \frac{1}{7}) + (\frac{1}{9} - \frac{1}{11}) + \cdots \cdots$
$= \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \cdots \cdots$

or, $\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots \cdots$

2. If $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, prove that

$$\tan^{-1}\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - \cdots \quad \cdots$$

$$\frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2}.$$

Therefore $0 \le \frac{1-\cos\theta}{1+\cos\theta} \le 1$ if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

So the expansion of $\tan^{-1}\frac{1-\cos\theta}{1+\cos\theta}$ in Gregory's series is valid when $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and

$$\tan^{-1}\frac{1-\cos\theta}{1+\cos\theta} = \tan^{-1}(\tan^2\tfrac{\theta}{2})$$
$$= \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - \cdots \quad \cdots$$

Exercises 2C

1. If z_1, z_2 be complex numbers, prove that

(i) $\sin z_1 + \sin z_2 = 2\sin\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2}$

(ii) $\cos z_1 + \cos z_2 = 2\cos\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2}$

(iii) $\tan(z_1+z_2) = \frac{\tan z_1 + \tan z_2}{1-\tan z_1 \tan z_2}$.

2. If n be an integer and z be a complex number, prove that

(i) $(\cosh z + \sinh z)^n = \cosh nz + \sinh nz$

(ii) $(1+\cosh 2z + \sinh 2z)^n = 2^n \cosh^n(\cosh nz + \sinh nz)$.

3. If x, y be real numbers, prove that

(i) $\tan(x+iy) = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$

(ii) $\cot(x+iy) = \frac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x}$

(iii) $\tanh(x+iy) = \frac{\sinh 2x + i\sin 2y}{\cosh 2x + \cos 2y}$.

4. Show that

(i) $\cos[i\log(2+\sqrt{3})] = 2$, (ii) $\sin[i\log(1+\sqrt{2})] = i$,

(iii) $\tan[\frac{\pi}{2} + \frac{1}{2}i\log 3] = 2i$, (iv) $\tan[\frac{1}{2}i\log 3] = \frac{1}{2}$,

(v) $\cosh[\log(2+\sqrt{3})] = 2$.

5. If $\sin(\theta + i\phi) = \tan\beta + i\sec\beta$, prove that

$\cos 2\theta \cosh 2\phi = 3$.

6. If $\tan(\theta + i\phi) = \sin(\alpha + i\beta)$, prove that

$\sin 2\theta \cot\alpha = \sinh 2\phi \coth\beta$.

7. If $\log\sin(x+iy) = u + iv \quad (0 < x < \pi)$, prove that

(i) $u = \frac{1}{2}\log(\cosh^2 y - \cos^2 x)$

(ii) $v = \tan^{-1}(\cot x \tanh y)$.

[**Hint.** Let $\sin x\cosh y = r\cos\theta, \cos x\sinh y = r\sin\theta$, where $-\pi < \theta \le \pi$. Then $r^2 = \cosh^2 y - \cos^2 x$, $\tan\theta = \cot x\tanh y$. $0 < x < \pi \Rightarrow \sin x > 0 \Rightarrow \cos\theta > 0$. Therefore $\theta = \tan^{-1}(\coth x \tanh y)$.]

8. Find the general solution of

(i) $\sin z = \frac{1}{2}$, (ii) $\cos z = \frac{1}{2}$,
(iii) $\sin z = 2i$, (iv) $\cos z = 2i$,
(v) $\sin z = -2$, (vi) $\cos z = -2$,
(vii) $\sinh z = 2$, (viii) $\cosh z = 2$.

9. Show that

(i) $\mathrm{Tan}^{-1}(1) = n\pi + \frac{\pi}{4}$, n being an integer

(ii) $\mathrm{Tan}^{-1}(-1) = n\pi - \frac{\pi}{4}$, n being an integer

(iii) $\mathrm{Tan}^{-1}(1+i) = \frac{1}{2}[(2n+1)\pi + \tan^{-1}(-2)] + \frac{i}{4}\log 5$, n being an integer

(iv) $\mathrm{Tan}^{-1}(-1+i) = \frac{1}{2}[(2n+1)\pi + \tan^{-1}2] + \frac{i}{4}\log 5$, n being an integer.

10. If x be a real number, prove that

(i) $\mathrm{Sin}^{-1}(ix) = n\pi + i(-1)^n\log(x + \sqrt{x^2+1})$, n being an integer

(ii) $\mathrm{Cos}^{-1}(ix) = 2n\pi \pm [\frac{\pi}{2} - i\log(x+\sqrt{x^2+1})]$, n being an integer

(iii) $\mathrm{Tan}^{-1}(ix) = n\pi + \frac{i}{2}\log(\frac{1+x}{1-x}), \quad -1 < x < 1$

$$= n\pi + \frac{\pi}{2} + \frac{i}{2}\log(\frac{x+1}{x-1}), \quad x > 1 \text{ or } x < -1.$$

[**Hint.** $\mathrm{Tan}^{-1}(ix) = -\frac{1}{2}i\,\mathrm{Log}\frac{1-x}{1+x}$.

$|x| < 1 \Rightarrow \frac{1-x}{1+x} = \frac{1-x^2}{(1+x)^2} > 0 \Rightarrow \mathrm{Log}\frac{1-x}{1+x} = \log\frac{1-x}{1+x} + 2n\pi i$.

$|x| > 1 \Rightarrow \frac{1-x}{1+x} = \frac{1-x^2}{(1+x)^2} < 0 \Rightarrow \mathrm{Log}\frac{1-x}{1+x} = \log\frac{x-1}{x+1} + (2n\pi+\pi)i$.]

11. If x be a real number > 1, prove that

(i) $\mathrm{Sin}^{-1}x = n\pi + (-1)^n[\frac{\pi}{2} - i\log(x+\sqrt{x^2-1})]$, n being an integer

(ii) $\mathrm{Cos}^{-1}x = 2n\pi \pm i\log(x+\sqrt{x^2-1})$, n being an integer

(iii) $\mathrm{Sin}^{-1}(-x) = 2n\pi - \frac{\pi}{2} \pm i\log(x-\sqrt{x^2-1})$, n being an integer

(iv) $\mathrm{Cos}^{-1}(-x) = (2n+1)\pi \pm i\log(x-\sqrt{x^2-1})$, n being an integer.

[**Hint.** (iii) Let $\text{Sin}^{-1}(-x) = u + iv$. Then $\sin u \cosh v = -x, \cos u \sinh v = 0$. $\sinh v = 0 \Rightarrow \cosh v = \pm 1 \Rightarrow \sin u = \pm x$, an impossibility. $\cos u = 0 \Rightarrow u = 2n\pi \pm \frac{\pi}{2}$, $u = 2n\pi + \frac{\pi}{2}$ and $\sin u \cosh v = -x \Rightarrow \cosh v = -x$, an impossibility. So $u = 2n\pi - \frac{\pi}{2}$ and $\cosh v = x$.]

12. If $\cos^{-1}(u + iv) = p + iq$ where u, v, p, q are real, prove that $\cos^2 p$ and $\cosh^2 q$ are the roots of the equation

$$x^2 - x(1 + u^2 + v^2) + u^2 = 0.$$

13. If $\sin^{-1}(u + iv) = p + iq$ where u, v, p, q are real, prove that $\sin^2 p$ and $\cosh^2 q$ are the roots of the equation

$$x^2 - x(1 + u^2 + v^2) + u^2 = 0.$$

14. (i) If $|\cos(x + iy)| = 1$, prove that $\cos 2x + \cosh 2y = 2$.

(ii) If $|\sin(x + iy)| = 1$, prove that $\cosh 2y - \cos 2x = 2$.

15. Prove that the general solution of

(i) $\sin z = \cosh 4$ is $(4n + 1)\frac{\pi}{2} \pm 4i$, n being an integer,

(ii) $\cos z = \cosh 4$ is $2n\pi \pm 4i$, n being an integer,

(iii) $\tan z = 1 + \sqrt{2}i$ is $(8n + 3)\frac{\pi}{8} + \frac{i}{2}\log(\sqrt{2} + 1)$, n being an integer.

[**Hint.** (iii) $\tan z = 1 + \sqrt{2}i \Rightarrow \exp(2iz) = \frac{-1+i}{2+\sqrt{2}} = \frac{1}{\sqrt{2}+1}[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}]$.]

16. If $x + iy = c \sin(u + iv)$ prove that when u = constant, the point (x, y) lies on a family of confocal hyperbolas;and when v = constant, the point (x, y) lies on a family of confocal ellipses, c being a parameter in both the cases.

17. Prove that

(i) $\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \cdots \quad \cdots$

(ii) $\frac{\pi}{12} = (1 - \frac{1}{\sqrt{3}}) - \frac{1}{3}(1 - \frac{1}{3\sqrt{3}}) + \frac{1}{5}(1 - \frac{1}{3^2\sqrt{3}}) - \cdots \quad \cdots$

(iii) $\frac{\pi}{4} = (\frac{1}{2} + \frac{1}{3}) - \frac{1}{3}(\frac{1}{2^3} + \frac{1}{3^3}) + \frac{1}{5}(\frac{1}{2^5} + \frac{1}{3^5}) - \cdots \quad \cdots$

18. If α, β, γ be three cube roots of unity, prove that

$$\frac{\tan^{-1}\alpha}{\alpha} + \frac{\tan^{-1}\beta}{\beta} + \frac{\tan^{-1}\gamma}{\gamma} = 3(1 - \frac{1}{7} + \frac{1}{13} - \frac{1}{19} + \cdots \quad \cdots).$$

19. If $0 \le \theta < \frac{\pi}{2}$ prove that the principal value of $\tan^{-1}(\cos\theta + i \sin\theta)$ is $\frac{\pi}{4} + \frac{i}{2}\log\tan(\frac{\pi}{4} + \frac{\theta}{2})$.

Deduce that when $0 \le \theta < \frac{\pi}{2}$

(i) $\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \cdots \quad \cdots = \frac{\pi}{4}$,

(ii) $\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \cdots \quad \cdots = \frac{1}{2}\log\tan(\frac{\pi}{4} + \frac{\theta}{2})$.

3. INTEGERS

3.1. Natural numbers.

The set $\mathbb{N}$ consiting of numbers 1, 2, 3,... is called the *set of all natural numbers*. The *well ordering property* of the set $\mathbb{N}$ states that

every non-empty subset of $\mathbb{N}$ *contains a least element.*

This means that if S be a non-empty subset of $\mathbb{N}$ there is some natural number a in S such that $a \le x$ for all x in S.

3.1.1. Principle of induction.

Let S be a subset of $\mathbb{N}$ with the properties –

(i) 1 belongs to S, and

(ii) whenever a natural number k belongs to S, then $k + 1$ belongs to S.

Then $S = \mathbb{N}$.

Proof. Let T be the set of all those natural numbers which are not in S. The theorem will be proved if we can prove that T is an empty set.

Let us assume that T is a non-empty set. Then by the well ordering property T possesses a least element, say m. Since $1 \in S, m > 1$ and so $m - 1$ is a natural number. Again since m is the least element in $T, m - 1$ is not in T and so $m - 1$ is in S.

Since $m - 1$ is in S, by (ii) $(m - 1) + 1$ is in S, i.e., m is in S which is a contradiction.

Therefore our assumption is wrong and T is empty and the theorem is proved.

Theorem 3.1.2. Let E_n be a statement involving a natural number n. If

(i) E_1 is true, and

(ii) E_{k+1} is true whenever E_k is true, where k is a natural number, then E_n is true for all natural numbers.

Proof. Let S be the set of those natural numbers n for which the statement E_n is true.

Then S has the properties –
(i) $1 \in S$, and
(ii) $k+1 \in S$ whenever $k \in S$.

Then by the principle of induction $S = \mathbf{N}$.

Thus E_n is true for all $n \in \mathbf{N}$.

Note. To establish a theorem (or a proposition) involving natural numbers by the principle of induction, both the conditions (i) and (ii) must be established.

The condition (i) is called the *basis of induction* and the assumption made in the condition (ii) is called the *induction hypothesis*.

Worked Examples.

1. Use the principle of induction to prove that
$1+2+\cdots+n = \frac{n(n+1)}{2}$, for all natural numbers n.

Step 1. For $n = 1$ the statement is true because $1 = \frac{1(1+1)}{2}$.

Step 2. Let us assume that the statement is true for some natural number k. Then $1+2+\cdots+k = \frac{k(k+1)}{2}$.

Therefore $1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$.

This shows that the statement is true for the natural number $k+1$ if it is true for k.

By the principle of induction, the statement is true for all natural numbers n.

2. Prove that $3^{2n} - 8n - 1$ is divisible by 64 for all $n \in \mathbf{N}$.

We use the principle of induction to prove the statement. Let $f(n) = 3^{2n} - 8n - 1$.

Step 1. $f(1) = 9 - 8 - 1 = 0$. $f(1)$ is divisible by 64. Therefore the statement is true for $n = 1$.

Step 2.
$$\begin{aligned} f(k+1) - f(k) &= [3^{2k+2} - 8(k+1) - 1] - [3^{2k} - 8k - 1] \\ &= 8(3^{2k} - 1) = 8(9^k - 1) \\ &= 8.8(9^{k-1} + 9^{k-2} + \cdots + 1) \\ &= 64p \text{ where } p \text{ is an integer.} \end{aligned}$$

Therefore $f(k+1)$ is divisible by 64 if $f(k)$ is so.
This proves that the statement is true for $k+1$ if it is true for k.

By the principle of induction, the statement is true for all natural numbers n.

3. Use the principle of induction to prove that for all natural numbers n, $(a_1 a_2 \ldots a_{2^n})^{\frac{1}{2^n}} \leq \frac{a_1+a_2+\cdots+a_{2^n}}{2^n}$, where a_i's are positive real numbers for $i = 1, 2, \ldots, 2^n$.

The statement is true for $n = 1$, since $(a_1 a_2)^{\frac{1}{2}} \leq \frac{a_1+a_2}{2}$ $\cdots$ (i)

Let us assume that the statement is true for $n = k$, where k is a natural number.
Then $(a_1 a_2 \ldots a_{2^k})^{\frac{1}{2^k}} \leq \frac{a_1+a_2+\cdots+a_{2^k}}{2^k} = p$, say.
Let $b_i = a_{2^k+i}$ for $i = 1, 2, \ldots, 2^k$.
Then $(b_1 b_2 \ldots b_{2^k})^{\frac{1}{2^k}} \leq \frac{b_1+b_2+\cdots+b_{2^k}}{2^k} = q$, say.

$$\begin{aligned} \{(a_1 a_2 \ldots a_{2^k})^{\frac{1}{2^k}} (b_1 b_2 \ldots b_{2^k})^{\frac{1}{2^k}}\}^{\frac{1}{2}} &= (pq)^{\frac{1}{2}} \\ &\leq \frac{p+q}{2} \quad \cdots \text{ by (i)} \end{aligned}$$

or, $(a_1 a_2 \ldots a_{2^{k+1}})^{\frac{1}{2^{k+1}}} \leq \frac{(a_1+a_2+\cdots+a_{2^k})+(b_1+b_2+\cdots+b_{2^k})}{2^{k+1}}$

i.e., $(a_1 a_2 \ldots a_{2^{k+1}})^{\frac{1}{2^{k+1}}} \leq \frac{(a_1+a_2+\cdots+a_{2^{k+1}})}{2^{k+1}}$.

This shows that the statement is true for $k+1$, if it be true for k.

By the principle of induction, the statement is true for all $n \in \mathbf{N}$.

There is a variation of the principle of induction.

Let S be a non-empty subset of $\mathbf{N}$ such that

(i) $n_0 \in S$, and (ii) $k(\geq n_0) \in S$ implies $k+1 \in S$.
Then $S = \{n \in \mathbf{N} : n \geq n_0\}$.

We can utilise this principle to prove that if $P(n)$ be a statement involving a natural number n satisfying the conditions –

(i) $P(n_0)$ is true (n_0 being the least possible natural number) and (ii) for $k \geq n_0$, $P(k+1)$ is true whenever $P(k)$ is true,

then $P(n)$ is true for all $n \geq n_0$.

Worked Example (continued).

4. Prove that $n! > 2^n$ for all natural numbers $n \geq 4$.

Let $P(n)$ be the statement $n! > 2^n$.
The statements $P(1), P(2)$ and $P(3)$ are not true.
The statement $P(4)$ is true, since $4! > 2^4$.

Let us assume that $P(k)$ is true where k is a natural number ≥ 4. Then $k! > 2^k$.
Therefore $(k+1)! > 2^k.(k+1) > 2^{k+1}$, since $k+1 > 2$.

This shows that $P(k+1)$ is true whenever $P(k)$ is true.

Since the statement $P(n)$ is true for $n = 4$ (the least possible natural number), by the principle of induction the statement $P(n)$ is true for all natural numbers $n \geq 4$.

3.1.3. Second principle of induction.

Let S be a subset of $\mathbb{N}$ such that

(i) $1 \in S$, and (ii) if $\{1, 2, \ldots, k\} \subset S$, then $k+1 \in S$.
Then $S = \mathbb{N}$.

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$. If not, T being a non-empty subset of $\mathbb{N}$ must have a least element, say m, by the well ordering property of $\mathbb{N}$.

Since $1 \in S, m \neq 1$. Therefore $m > 1$.

By the choice of m, all natural numbers less than m belongs to S. Hence $1, 2, \ldots, m-1 \in S$.

By (ii) $m \in S$, a contradiction.

This proves $T = \phi$ and therefore $S = \mathbb{N}$.

Worked Example (continued).

5. Prove that for all $n \in \mathbb{N}$, $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an even integer.

Let $P(n)$ be the statement $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an even integer.

The statement $P(1)$ is true, since $(2+\sqrt{3})^1 + (2-\sqrt{3})^1 = 4$ and it is an even integer.

Let assume that $P(n)$ is true for $n = 1, 2, \ldots, k$.

$(2+\sqrt{3})^{k+1} + (2-\sqrt{3})^{k+1}$

$= a^{k+1} + b^{k+1}$, where $a = 2+\sqrt{3}, b = 2-\sqrt{3}$

$= (a^k + b^k)(a+b) - (a^{k-1} + b^{k-1})ab$

$= 4(a^k + b^k) - (a^{k-1} + b^{k-1})$.

This is an even integer, since $a^k + b^k$ and $a^{k-1} + b^{k-1}$ are even integers, by assumption.

This shows that $P(k+1)$ is true whenever $P(1), P(2), \ldots, P(k)$ are true.

By the second principle of induction, the statement $P(n)$ is true for all natural numbers n.

3.2. Integers.

The set of all integers, denoted by $\mathbb{Z}$, consists of whole numbers $0, \pm 1, \pm 2, \pm 3, \ldots$ ... The set of all positive integers (a proper subset of $\mathbb{Z}$) is identified with the set $\mathbb{N}$. We shall use the properties and principles of $\mathbb{N}$ in connection with the proof of any theorem about positive integers.

Theorem 3.2.1. Division algorithm.

Given integers a and b with $b > 0$, there exist unique integers q and r such that $a = bq + r$, where $0 \leq r < b$.

Proof. Let us consider the subset of integers

$$S = \{a - bx : x \in \mathbb{Z}, a - bx \geq 0\}.$$

First we show that S is non-empty.

Since $b \geq 1, |a| b \geq |a|$. Therefore $a + |a| b \geq a + |a| \geq 0$.

This shows that $a - b(-|a|) \in S$ and therefore S is non-empty.

Since S is a non-empty set of non-negative integers, either

(i) S contains 0 as its least element, or

(ii) S contains a smallest positive inetger as its least element by the well ordering property of the set $\mathbb{N}$.

In either case, we call it r. Therefore there exists an integer q such that $a - bq = r, r \geq 0$.

We assert that $r < b$. Because if $r \geq b$, then

$$a - (q+1)b = (a - qb) - b = r - b \geq 0.$$

This shows that $a - (q+1)b$ belongs to S and also $a - (q+1)b = r - b < r$. This leads to a contradiction to the fact that r is the least element in S.

Hence $r < b$ and consequently, $a = bq + r$ where $0 \leq r < b$.

In order to establish uniqueness of q and r, let us suppose that a has two representations: $a = bq + r$, $a = bq_1 + r_1$ where $0 \leq r < b$, $0 \leq r_1 < b$.

Then $b(q - q_1) = r_1 - r$ or, $b|q - q_1| = |r_1 - r|$.

But $0 \leq r_1 < b$ and $-b < -r \leq 0$ yield $-b < r_1 - r < b$, i.e., $|r_1 - r| < b$. Consequently, $|q - q_1| < 1$.

Since q and q_1 are integers, the only possibility is $q = q_1$ and therefore $r = r_1$.

This completes the proof.

Definition. q is called the *quotient* and r is called the *remainder* in the division of a by b.

A more general version of the Division algorithm is obtained by taking b a non-zero integer.

Theorem 3.2.2. Given integers a and b, with $b \neq 0$, there exist unique integers q and r such that $a = bq + r$, $0 \leq r < |b|$.

Proof. With the previous theorem already established, it is enough to consider the case in which b is negative. Then $|b| > 0$. By the previous theorem, there exist unique integers q_1 and r such that

$$\begin{aligned} a &= |b|q_1 + r, 0 \leq r < |b| \\ &= -bq_1 + r. \end{aligned}$$

Therefore $a = bq + r$ where $q = -q_1$.

To illustrate the division algorithm, let us take $b = 3, a = -20, 2, 10$.

$$\begin{aligned} -20 &= 3.-7+1 \text{ gives} && q = -7, r = 1 \\ 2 &= 3.0+2 \text{ gives} && q = 0, r = 2 \\ 10 &= 3.3+1 \text{ gives} && q = 3, r = 1. \end{aligned}$$

Let us take $b = -3, a = -20, 2, 10$.

$$\begin{aligned} -20 &= -3.7+1 \text{ gives} && q = 7, r = 1 \\ 2 &= -3.0+2 \text{ gives} && q = 0, r = 2 \\ 10 &= -3.-3+1 \text{ gives} && q = -3, r = 1. \end{aligned}$$

When the remainder in the division algorithm turns out to be 0, the case is of special interest to us.

Definition. An integer a is said to be *divisible* by an integer $b \neq 0$ if there exists some integer c such that $a = bc$.

We express this in symbol $b|a$ and read "b divides a". We also express this by the statements– "b is a divisor of a", "a is a multiple of b"

If b is a divisor of a, then $-b$ is also a divisor of a, because $a = bc \Rightarrow a = (-b)(-c)$. Thus divisors of an integer occur in pairs.

Note. In the symbol $b|a$, b is not 0 but a may be 0. Thus 0 is a multiple of every non-zero integer.

The following properties are immediate (assuming that a divisor is always a non-zero integer).

(i) $a \mid b$ and $b \mid c \Rightarrow a \mid c$,

(ii) $a \mid b$ and $b \mid a$ if and only if $a = \pm b$.

Theorem 3.2.3. If $a \mid b$ and $a \mid c$ then $a \mid (bx + cy)$ for arbitrary integers x and y.

Proof. Since $a \mid b$, $b = ad$ for some integer d.
Since $a \mid c$, $c = ae$ for some integer e.
Therefore $bx + cy = adx + aey = a(dx + ey)$.
This shows that $a \mid (bx + cy)$ whatever integers x, y may be.

Worked Examples.

1. Prove that the product of any m consecutive integers is divisible by m.

Let the consecutive integers be $c, c+1, c+2, \ldots, c+(m-1)$.
By division algorithm, there exist integers q and r such that $c = mq + r$, $0 \leq r < m$.
When $r = 0$, $c = mq$ and therefore $m \mid c$;
when $r = 1$, $c + (m-1) = m(q+1)$ and therefore $m \mid c + (m-1)$;
when $r = 2$, $c + m - 2 = m(q+1)$ and therefore $m \mid c + (m-2)$;
... ...
when $r = m-1$, $c + 1 = m(q+1)$ and therefore $m \mid c + 1$.

Therefore whatever integer r may be, m divides one of the integers $c, c+1, \ldots, c+(m-1)$ and it follows that the product $c(c+1)(c+2)\ldots(c+m-1)$ is always divisible by m.

2. Use division algorithm to prove that the square of an odd integer is of the form $8k + 1$, where k is an integer.

By division algorithm every integer, upon division by 4, leaves one of the remainders $0, 1, 2, 3$. Therefore any integer is one of the forms $4q, 4q+1, 4q+2, 4q+3$, where q is an integer.
Odd integers are of the forms $4q + 1, 4q + 3$.

$$\begin{aligned} \text{Now } (4q+1)^2 &= 8(2q^2+q)+1 \text{ is of the form } 8k+1, \\ (4q+3)^2 &= 8(2q^2+3q+1)+1 \text{ is of the form } 8k+1. \end{aligned}$$

Hence the square of an odd integer is of the form $8k + 1$.

Definition. If a and b are integers then an integer d is said to be a *common divisor* of a and b if $d \mid a$ as well as $d \mid b$.

Since 1 is a divisor of every integer, 1 is a common divisor of a and b. Therefore, for an arbitrary pair of integers a, b there exists always a common divisor.

If both of a and b be 0 then each integer is a common divisor of a and b. But if at least one of a and b is non-zero there is only a finite number of positive common divisors. Of these positive common divisors, there

is a greatest one, called the greatest common divisor and is denoted by $gcd(a, b)$.

Definition. If a and b are integers, not both zero, the *greatest common divisor* of a and b, denoted by $gcd(a, b)$ is the *positive integer* d satisfying

(i) $d \mid a$ and $d \mid b$; and

(ii) if $c \mid a$ and $c \mid b$ then $c \mid d$.

For example, let $a = 12, b = -18$. Then the positive divisors of 12 are $1, 2, 3, 4, 6, 12$ and those of -18 are $1, 2, 3, 6, 9, 18$. Therefore the positive common divisors are $1, 2, 3, 6$ and $gcd(12, -18) = 6$.

Note. It follows from the definition that $gcd(a, -b) = gcd(-a, b) = gcd(-a, -b) = gcd(a, b)$, where a, b are integers, not both zero.

Theorem 3.2.4. If a and b are integers, not both zero, then there exist integers u and v such that $gcd(a, b) = au + bv$.

Proof. Let $S = \{ax + by : x, y \in \mathbb{Z} \text{ and } ax + by > 0\}$. First we show that S is a non-empty set.

Since at least one of a, b is non-zero, let $a \neq 0$. Then $| a | > 0$.

Therefore $| a | = a.x + b.0$ is an element of S, where we choose $x = 1$ if $a > 0$ and $x = -1$ if $a < 0$.

Since S is a non-empty set of positive integers, by the well ordering property of the set $\mathbb{N}$, S contains a least element, say d.

Then $d = au + bv$ for some integers u, v.

By division algorithm, $a = dq + r$ where q and r are integers with $0 \leq r < d$.

$$\begin{aligned} \text{Therefore } r &= a - dq \\ &= a - (au + bv)q \\ &= a(1 - uq) + b(-vq). \end{aligned}$$

This representation shows that if $r > 0$ then $r \in S$.

But d is the least element in S and since $r < d$, $r \notin S$.

Consequently, $r = 0$. This proves that $a = dq$, i.e., d is a divisor of a.

By similar arguments we can prove that d is a divisor of b.

Therefore d becomes a common divisor of a and b.

To prove that d is the $gcd(a, b)$, let us assume that c is a common divisor of a and b.

Then $c \mid a$ and $c \mid b$ and therefore $c \mid (au + bv)$, by Theorem 3.2.3 i.e., $c \mid d$ and consequently, d is the greatest common divisor.

This completes the proof.

For example,

$gcd(-4, 20) = 4$ and $4 = -4.(-1) + 20.0$

$gcd(55, 35) = 5$ and $5 = 55.2 + 35.(-3)$

$gcd(0, 9) = 9$ and $9 = 0.0 + 9.1$

$gcd(-9, 13) = 1$ and $1 = -9.(-3) + 13. - 2.$

Note 1. The $gcd(a, b)$ is the *least positive* value of $ax + by$ where x, y are integers.

But x and y are not uniquely determined integers for which the integer $ax + by$ is least positive. Because if $d = au + bv$, where u and v are integers then d can also be expressed as $d = a(u + kb) + b(v - ka)$ where k is an integer.

For example, let $a = 15, b = 24$. Then $d = 3$ and $d = 15(-3) + 24.2$ which can also be expressed as $d = 15.(-3 + 24k) + 24(2 - 15k)$ for any integer k.

Note 2. Guaranteed by the theorem it is always possible to express $gcd(a, b)$ as a linear combination of a and b. But the theorem gives no clue how to express $gcd(a, b)$ in the desired form $au + bv$, i.e., how to determine u and v. This will be discussed in a subsequent article.

Worked Example (continued).

3. Show that $gcd(a, a + 2) = 1$ or 2 for every integer a.

Let $d = gcd(a, a + 2)$. Then $d \mid a$ and $d \mid a + 2$.

Therefore $d \mid ax + (a + 2)y$ for all integers x, y.

Taking $x = -1$, $y = 1$, it follows that $d \mid 2$. i.e., d is either 1 or 2.

Theorem 3.2.5. If k be a positive integer, $gcd(ka, kb) = k.gcd(a, b)$.

Proof. Let $d = gcd(a, b)$. Then there exist integers u and v such that $d = au + bv$. Since $d = gcd(a, b)$, $d \mid a$ and $d \mid b$.

$d \mid a \Rightarrow kd \mid ka, d \mid b \Rightarrow kd \mid kb$.

Therefore kd is a common divisor of ka and kb.

Let c be a common divisor of ka and kb.

$c \mid ka \Rightarrow ka = pc$ for some integer p; and $c \mid kb \Rightarrow kb = qc$ for some integer q.

Now $kd = k(au + bv) = pcu + qcv = (pu + qv)c$.

As $pu + qv$ is an integer, it follows that $c \mid kd$.

Consequently, $kd = gcd(ka, kb)$, i.e., $gcd(ka, kb) = k.gcd(a, b)$.

This completes the proof.

Definition. Two integers a and b, not both zero, are said to be prime *to each other (or relatively prime)* if $gcd(a, b) = 1$.

Theorem 3.2.6. Let a and b be integers, not both zero. Then a and b are prime to each other if and only if there exist integers u and v such that $1 = au + bv$.

Proof. Let a and b be prime to each other. Then $gcd(a, b) = 1$. Therefore there exist integers u and v such that $1 = au + bv$.

Conversely, let us suppose that there are integers u and v such that $1 = au + bv$ and let $d = gcd(a, b)$.

Since $d \mid a$ and $d \mid b$ then $d \mid ax + by$ for all integers x and y.

Hence $d \mid 1$ and this implies $d = 1$, since d is a positive integer.

Theorem 3.2.7. If $d = gcd(a, b)$, then $\frac{a}{d}$ and $\frac{b}{d}$ are integers prime to each other.

Proof. Since $d \mid a$, there exists an integer m such that $md = a$.

Since $d \mid b$, there exists an integer n such that $nd = b$.

As $\frac{a}{d} = m$ and $\frac{b}{d} = n$, $\frac{a}{d}$ and $\frac{b}{d}$ are integers.

Since $d = gcd(a, b)$, it is possible to find integers u and v such that $d = au + bv$. Therefore $1 = (\frac{a}{d})u + (\frac{b}{d})v$.

This form of representation shows that $\frac{a}{d}$ and $\frac{b}{d}$ are integers prime to each other.

Theorem 3.2.8. If $a \mid bc$ and $gcd(a, b) = 1$, then $a \mid c$.

Proof. Since $gcd(a, b) = 1$, there exist integers u and v such that $1 = au + bv$. Therefore $c = acu + bcv$.

Since $a \mid ac$ and $a \mid bc$, it follows that $a \mid \{(ac)u + (bc)v\}$ which means $a \mid c$.

Another proof. $gcd(a, b) = 1 \Rightarrow gcd(ac, bc) = c$. $a \mid bc \Rightarrow a$ is a common divisor of ac and bc. $gcd(ac, bc) = c$ and a is a common divisor of ac and bc imply $a \mid c$.

Corollary. If $ap = bq$ and a is prime to b then $a \mid q$ and $b \mid p$.

Theorem 3.2.9. If $a \mid c$ and $b \mid c$ with $gcd(a, b) = 1$, then $ab \mid c$.

Proof. Since $a \mid c$ and $b \mid c$, there exist integers m and n such that $c = am = bn$.

Since $gcd(a, b) = 1$, there exist integers u, v such that $1 = au + vb$.

Therefore $c = (au)c + (bv)c$
$= ab(un + vm) \Rightarrow ab \mid c$.

Note. Without the condition $gcd(a, b) = 1$, $a|c$ and $b|c$ together may not imply $ab|c$.

For example, $4 \mid 12$ and $6 \mid 12$ do not imply $4.6 \mid 12$.

Theorem 3.2.10. If a is prime to b and a is prime to c then a is prime to bc.

Proof. Since a is prime to b, $au + bv = 1$ for some integers u, v ... (i)

Since a is prime to c, $am + cn = 1$ for some integers m, n ... (ii)

From (i) $acun + bcvn = cn = 1 - am$ by (ii).

or, $a(m + cun) + bc(vn) = 1$.

Since $m + cun$ and vn are integers, it follows that a is prime to bc.

Worked Examples (continued).

4. If a is prime to b, prove that $a + b$ is to prime to ab.

Since a is prime to b, there exist integers u and v such that $au + bv = 1$.

This can be expressed as $a(u - v) + (a + b)v = 1$.

Since $u - v$ and v are integers, it follows that a is prime to $a + b$.

Again, $au + bv = 1$ can be expressed as $(a + b)u + b(v - u) = 1$.

Since $v - u$ and u are integers, it follows that $a + b$ is prime to b.

By Theorem 3.2.10, $a + b$ is prime to ab.

5. If a is prime to b, prove that

(i) a^2 is prime to b,

(ii) a^2 is prime to b^2.

(i) Since a is prime to b, there exist integers u and v such that $au + bv = 1$. Then $au = 1 - bv$

or, $a^2u^2 = 1 - 2bv + b^2v^2$

or, $a^2u^2 + b(2v - bv^2) = 1$.

Since u^2 and $2v - bv^2$ are integers, it follows that a^2 is prime to b.

(ii) Since a^2 is prime to b, there exist integers m and n such that $a^2m + bn = 1$. Then $bn = 1 - a^2m$

or, $b^2n^2 = 1 - 2a^2m + a^4m^2$

or, $a^2(2m - a^2m^2) + b^2n^2 = 1$.

Since n^2 and $2m - a^2m^2$ are integers, it follows that a^2 is prime to b^2.

6. If $d = gcd(a, b)$, show that $gcd(a^2, b^2) = d^2$.

Since $d = gcd(a, b)$, $a = dp$ and $b = dq$, where p, q are integers prime to each other.

Therefore $a^2 = d^2p^2, b^2 = d^2q^2$ and this shows that d^2 is a common divisor of a^2 and b^2.

Let $gcd(a^2, b^2) = d^2u$, where u is a positive integer. Then $d^2u|d^2p^2$ and $d^2u|d^2q^2$ and therefore $u|p^2$ and $u|q^2$.

But $gcd(p, q) = 1 \Rightarrow gcd(p^2, q^2) = 1$.

Since u is a common divisor of p^2 and q^2 and $gcd(p^2, q^2) = 1$, it follows that $u = 1$. Hence $gcd(a^2, b^2) = d^2$.

7. If $gcd(a, b) = 1$, show that $gcd(a + b, a^2 - ab + b^2) = 1$ or 3.

Let $d = gcd(a + b, a^2 - ab + b^2)$. Then $d \mid a + b$ and $d \mid (a^2 - ab + b^2)$.
This implies $d \mid (a + b)(a + b) - (a^2 - ab + b^2)$, i.e., $d \mid 3ab$.

Therefore $d \mid a + b$ and $d \mid 3ab$. Since $gcd(a, b) = 1$, it follows that $gcd(a+b, ab) = 1$. There exist integers u and v such that $u(a+b)+v(ab) = 1$. Since $d \mid a+b$, $a+b = dp$ for some integer p. Therefore $(up)d+v(ab) = 1$ and this shows that d is prime to ab.

$d \mid 3ab$ and d is prime to ab implies $d \mid 3$. Therefore $d = 1$ or $d = 3$.

8. Prove that the product of any three consecutive integers is divisible by 6.

By division algorithm, any integer, upon division by 3, leaves one of the remainders $0, 1, 2$. Therefore any integer n is one of the forms $3k, 3k + 1, 3k + 2$.

When $n = 3k$, n is divisible by 3.
When $n = 3k + 1$, $n + 2$ is divisible by 3.
When $n = 3k + 2$, $n + 1$ is divisible by 3.

It follows that for any integer n, $n(n + 1)(n + 2)$ is divisible by 3.

Again, the product of two consecutive integers is divisible by 2.
Therefore $2 \mid n(n + 1)(n + 2)$ and $3 \mid n(n + 1)(n + 2)$.
Since $gcd(2, 3) = 1$, it follows that $2.3 \mid n(n + 1)(n + 2)$, i.e., $6 \mid n(n + 1)(n + 2)$.

9. Prove that $1 + 2 + \cdots + n$ is a divisor of $1^r + 2^r + \cdots + n^r$ for any odd positive integer r.

$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Let $S_n = 1^r + 2^r + \cdots + n^r$.
Then $2s_n = (1^r + n^r) + (2^r + (n - 1)^r) + \cdots + (n^r + 1^r)$... ... (i)

Since r is an odd positive integer, $n + 1$ is a divisor of each term in the right hand side of (i). Therefore $(n + 1)|2S_n$... ... (ii)

Again $2S_n = [1^r + (n - 1)^r] + [2^r + (n - 2)^r] + \cdots + [(n - 1)^r + 1^r] + 2n^r$ ... ... (iii)

Clearly, n is a divisor of each term in the right hand side of (iii).
Therefore $n|2S_n$... ... (iv)

Because n and $n + 1$ are prime to each other, it follows from (ii) and (iv) that $n(n + 1)$ is a divisor of $2S_n$.

As $n(n+1)$ is divisible by 2, $\frac{n(n+1)}{2}$ is an integer and therefore $\frac{n(n+1)}{2}$ is a divisor of S_n.

That is, $1 + 2 + \cdots + n$ is a divisor of $1^r + 2^r + \cdots + n^r$ if r is an odd positive integer.

3.2.11. Euclidean algorithm.

Euclidean algorithm is an efficient method of finding the greatest common divisor of two given integers. The method involves repeated application of the division algorithm.

Let a and b be two integers whose *g.c.d.* is required.

Since $gcd(a, b) = gcd(|a|, |b|)$, it is enough to assume that a and b are positive integers. Without loss of generality, we assume $a > b > 0$.

By division algorithm, $a = bq_1 + r_1$ where $0 \le r_1 < b$.
If it happens that $r_1 = 0$, then $b \mid a$ and $gcd(a, b) = b$.

If $r_1 \neq 0$, then by division algorithm, $b = r_1q_2 + r_2$ where $0 \le r_2 < r_1$.

If $r_2 = 0$, the process stops. If $r_2 \neq 0$, then by division algorithm,
$r = r_2q_3 + r_3$ where $0 \le r_3 < r_2$.

The process continues until some zero remainder appears. This must happen because the remainders $r_1, r_2, r_3, \ldots$ form a decreasing sequence of integers and since $r_1 < b$, the sequence contains at most b non-negative integers.

Let us assume that $r_{n+1} = 0$ and r_n is the last non-zero remainder.

We have the following relations

$$\begin{aligned}
a &= bq_1 + r_1 && 0 < r_1 < b;\\
b &= r_1q_2 + r_2 && 0 < r_2 < r_1;\\
r_1 &= r_2q_3 + r_3 && 0 < r_3 < r_2;\\
&\ldots \quad \ldots && \ldots \quad \ldots\\
r_{n-2} &= r_{n-1}q_n + r_n && 0 < r_n < r_{n-1};\\
r_{n-1} &= r_nq_{n+1} + 0.
\end{aligned}$$

We assert that r_n is the $gcd(a,b)$. First of all we prove the

lemma – If $a = bq + r$, then $gcd(a,b) = gcd(b,r)$.

Proof. Let $d = gcd(a,b)$. Then $d \mid a$ and $d \mid b$.

This implies $d|(a-bq)$, i.e., $d|r$. This shows that d is a common divisor of b and r.

Let c be a common divisor of b and r. Then $c \mid bq + r$, i.e., $c \mid a$.
This shows that c is a common divisor of a and b.
Since $d = gcd(a,b)$, it follows from the property of the g.c.d. that $c \mid d$ and this gives $d = gcd(b,r)$.

We utilise the lemma to show that $r_n = gcd(a,b)$.
$r_n = gcd(0, r_n) = gcd(r_{n-1}, r_n) = gcd(r_{n-2}, r_{n-1}) = \cdots \cdots = gcd(b, r_1) = gcd(a,b)$.

Also we have $r_n = r_{n-2} - r_{n-1}q_n$
$= r_{n-2} - (r_{n-3} - r_{n-2}q_{n-1})q_n$
$= (1 + q_{n-1}q_n)r_{n-2} + (-q_n)r_{n-3}$.

r_n is expressed as a linear combination of r_{n-2} and r_{n-3}. Proceeding backwards we can express r_n as a linear combination of a and b.

Worked Examples (continued).

10. Calculate $gcd(567, 315)$ and express $gcd(567, 315)$ as $567u + 315v$, where u and v are integers.

By division algorithm,
$\frac{567}{315} = 1 + \frac{252}{315}$, $\frac{315}{252} = 1 + \frac{63}{252}$, $\frac{252}{63} = 4$.
Then $567 = 315.1 + 252$, $315 = 252.1 + 63$, $252 = 63.4 + 0$.
The last non-zero remainder is 63. Therefore $gcd(567, 315) = 63$.

$$\begin{aligned} 63 = 315 - 252.1 &= 315 - (567 - 315) \\ &= 567.(-1) + 315.2 \\ &= 567u + 315v, \text{ where } u = -1, v = 2. \end{aligned}$$

11. Find two integers u and v satisfying $63u + 55v = 1$.

63 and 55 are integers prime to each other and therefore there exist integers u, v such that $63u + 55v = 1$.

By division algorithm,
$63 = 55.1 + 8$, $55 = 8.6 + 7$, $8 = 7.1 + 1$.
We have $1 = 8 - 7 = 8 - (55 - 8.6) = 8.7 - 55$
$= (63 - 55).7 - 55 = 63.7 + 55.(-8)$.
Therefore $u = 7, v = -8$.

12. Find two integers u and v satisfying $54u + 24v = 30$.

Let us find the $gcd(54, 24)$.
By division algorithm, $54 = 24.2 + 6$, $24 = 6.4 + 0$.
Therefore $gcd(54, 24) = 6$.
Now $6 = 54 - 24.2 = 54.1 + 24.(-2)$.
Consequently, $30 = 54.5 + 24.(-10)$. Therefore $u = 5, v = -10$.

Definition. Let $a_1, a_2, \ldots, a_n$ be integers different from zero. An integer b is said to be a *common multiple* of $a_1, a_2, \ldots, a_n$ if $a_i \mid b$ for $i = 1, 2, \ldots, n$.

In fact, common multiples do exist. 0 is a common multiple of $a_1, a_2, \ldots, a_n$. The products $a_1 a_2 \ldots a_n$ and $-a_1 a_2 \ldots a_n$ are both common multiples of $a_1, a_2, \ldots, a_n$ and one of these is positive. By the well ordering property of the set $\mathbb{N}$, the set of all positive common multiples contains a least element which is of special interest.

Definition. Let $a_1, a_2, \ldots, a_n$ be integers different from zero. The least positive common multiple is said to be the *least common multiple* (l.c.m.) of the integers $a_1, a_2, \ldots, a_n$ and it is denoted by $[a_1, a_2, \ldots, a_n]$.

For example, the l.c.m. of $2, 3, 6$ is 6; the l.c.m. of $-2, -3, -6$ is 6; the l.c.m. of $-2, -6, 10$ is 30.

Theorem 3.2.12. If h be any common multiple of the integers $a_1, a_2, \ldots, a_n$, none of which is zero, then $[a_1, a_2, \ldots, a_n] \mid h$.

Proof. Let m be the l.c.m. of $a_1, a_2, \ldots, a_n$. By division algorithm, there exist integers q and r such that $h = mq + r$, where $0 \le r < m$.

For each $i = 1, 2, \ldots, n$, $a_i \mid h$ and $a_i \mid m$ and therefore $a_i \mid r$.

This shows that r is a common multiple. If $r > 0$ then r becomes a positive common multiple less than the least common multiple m, a contradiction.

Therefore $r = 0$ and consequently, m is a divisor of h.
This completes the proof.

Note. If $m = [a_1, a_2, \cdots, a_n]$ then the common multiples of the integers $a_1, a_2, \ldots, a_n$ is the set $\{0, \pm m, \pm 2m, \cdots\}$.

Theorem 3.2.13. If a, b are integers different from zero and k is a positive integer, then $[ka, kb] = k[a, b]$.

Proof. Let $M = [ka, kb]$. Then M is a multiple of ka, kb.

Let $M = p(ka) = q(kb)$, where p, q are integers. Then $\frac{M}{k} = pa = qb$.
This shows that $\frac{M}{k}$ is a common multiple of a and b.

Let $m = [a, b]$. Then $m \leq \frac{M}{k}$, i.e., $M \geq mk$.

m is a multiple of a and b. Let $m = ar = bs$, where r, s are integers.

Then $mk = (ak)r = (bk)s$. This shows that mk is a common multiple of ak and bk. Since $M = [ka, kb]$, $M \leq mk$.

Consequently, $M = mk$, i.e., $[ka, kb] = k[a, b]$.

Theorem 3.2.14. If a, b are integers different from zero, then

$$[a, b](a, b) = |\, ab \,|,$$

where $[a, b]$ = the l.c.m. of a and b; (a, b) = the *g.c.d.* of a and b.

Proof. First we assume that a and b are positive integers.

Case 1. Let $(a, b) = 1$.

As $[a, b]$ is a multiple of a and b, let $[a, b] = pa$ for some integer p. Then $b \mid pa$.

Since $(a, b) = 1$, it follows that $b \mid p$. Therefore $b \leq p, ba \leq pa$.

Again, ba being a common multiple of a and b cannot be less than $[a, b]$, i.e., cannot be less than pa, so that $ba = pa$.

Thus $[a, b] = ab$ and this gives $[a, b](a, b) = ab$.

Case 2. Let $(a, b) = d > 1$. Then $(\frac{a}{b}, \frac{b}{d}) = 1$, by theorem 3.2.7.

Since $(\frac{a}{b}, \frac{b}{d}) = 1$, we obtain, by case 1, $[\frac{a}{d}, \frac{b}{d}](\frac{a}{d}, \frac{b}{d}) = \frac{a}{d}\frac{b}{d}$

or, $d[\frac{a}{d}, \frac{b}{d}]d(\frac{a}{d}, \frac{b}{d}) = ab$.

But $d[\frac{a}{d}, \frac{b}{d}] = [a, b]$, by theorem 3.2.13 and $d(\frac{a}{d}, \frac{b}{d}) = (a, b)$, by theorem 3.2.5. Therefore $[a, b](a, b) = ab$.

In the general case, let a, b be non-zero integers, positive or negative.

Since $[a, -b] = [-a, b] = [-a, -b] = [a, b], (a, -b) = (-a, b) = (-a, -b) = (a, b)$, whatever non-zero integers a, b may be, it follows that $[a, b] = [|\, a \,|, |\, b \,|], (a, b) = (|\, a \,|, |\, b \,|)$.

Hence $[a, b](a, b) = [|\, a \,|, |\, b \,|](|\, a \,|, |\, b \,|)$

$= |\, a \,||\, b \,|$, by what we have proved

$= |\, ab \,|$.

This completes the proof.

Linear Diophantine equation.

An equation in one or more unknowns which is to be solved in integers is said to be a *Diophantine equation*, named after the Greek mathematician Diophantus, who initiated the study of such problems.

A linear Diophantine equation of the form $ax + by = c$ may have many solutions in integers or may not have even a single solution.

For example, the equation $2x + 4y = 6$ has many solutions in integers, since $2.1 + 4.1 = 6$, $2.5 + 4.(-1) = 6$, $2.9 + 4.(-3) = 6$,...

Whereas, the equation $2x + 4y = 3$ cannot have a solution in integers, since the left hand side is always an even integer for every pair of integers x and y, while the right hand side is odd.

First of all, we discuss the condition for solvability of the linear equation $ax + by = c$ in integers, where a, b, c are integers and a, b are not both zero.

Theorem 3.2.15. If a, b, c are integers and a, b are not both zero, the equation $ax + by = c$ has an integral solution if and only if d is a divisor of c, where $d = gcd(a, b)$. If (x_0, y_0) be any particular solution of the equation, then all integral solutions are given by $(x_0 + \frac{b}{d}t, y_0 - \frac{a}{d}t)$ for different integers t.

Proof. Let (x_1, y_1) be an integral solution of the equation $ax + by = c$.

Then $ax_1 + by_1 = c$, where x_1, y_1 are integers.

Let $gcd(a, b) = d$. Then $d \mid a$ and $d \mid b$.

This implies $d \mid ax_1 + by_1$, i.e., $d \mid c$.

Conversely, let $gcd(a, b)$ be a divisor of c.

Let $gcd(a, b) = d$. Then $d = au + bv$ for some integers u, v.

Let $c = dp$ where p is an integer.

Then $c = (au + bv)p = a(up) + b(vp)$.

This shows that (up, vp) is a solution of the equation $ax + by = c$. Clearly, up and vp are integers. So the equation $ax + by = c$ has an integral solution.

To prove the second part, let (x', y') be any other solution. Then $ax_0 + by_0 = c = ax' + by'$, which gives $a(x' - x_0) = b(y_0 - y')$.

Since $d = gcd(a, b)$, there exist relatively prime integers p, q such that $a = dp$ and $b = dq$. Therefore we have $p(x' - x_0) = q(y_0 - y')$.

This shows that $p \mid q(y_0 - y')$ with $gcd(p, q) = 1$ and therefore $p \mid (y_0 - y')$. Therefore $y_0 - y' = pt$ for some integer t. Also we have $x' - x_0 = qt$.

This gives $x' = x_0 + qt = x_0 + \frac{b}{d}t$, $y' = y_0 - pt = y_0 - \frac{a}{d}t$.

Thus there are infinite number of solutions, one for each integral value of t.

Note. In particular, if a and b are prime to each other then all integral solutions of the equation are given by

$$x = x_0 + bt, y = y_0 - at \text{ for all integral values of } t.$$

3.2.16. Integral solution of the equation $ax + by = c$, where a, b, c are positive integers and $gcd(a, b) = 1$.

Since $gcd(a, b) = 1$, there exist integers u and v such that $au + bv = 1$.
Therefore $ax + by = c(au + bv)$
or, $a(x - cu) = -b(y - cv)$.

Since a and b are prime to each other, $x - cu$ is divisible by b and $y - cv$ is divisible by a and therefore

$\frac{x-cu}{-b} = \frac{y-cv}{a} = t$, where t is an integer
or, $x = cu - bt$
$y = cv + at$, where $t = 0, \pm 1, \pm 2, \cdots$

This is the general solution in integers.

Note. For positive integral solution, we must have $cu - bt > 0$ and $cv + at > 0$ simultaneously. Hence $\frac{-cv}{a} < t < \frac{cu}{b}$.

If $\frac{cu}{b} = m + f$ where m is an integer and $0 < f \leq 1$, then $t \leq m$.

If $\frac{-cv}{a} = n + f'$ where n is an integer and $0 \leq f' < 1$, then $t > n$.

The total number of solutions in positive integers is $m - n$.

3.2.17. Integral solution of the equation $ax - by = c$, where a, b, c are positive integers and $gcd(a, b) = 1$.

Since $gcd(a, b) = 1$, there exist integers u and v such that $au + bv = 1$.
Therefore $ax - by = c(au + bv)$
or, $a(x - cu) = b(y + cv)$.

Since a and b are prime to each other, $x - cu$ is divisible by b and $y + cv$ is divisible by a and therefore

$\frac{x-cu}{b} = \frac{y+cv}{a} = t$, where t is an integer
or, $x = cu + bt$
$y = -cv + at$, where $t = 0, \pm 1, \pm 2, \cdots$

This is the general solution in integers.

Note. For a positive integral solution, we must have $cu + bt > 0$ and $-cv + at > 0$ simultaneously. Hence $t > \frac{-cu}{b}$ and $t > \frac{cv}{a}$.

Let the integral part of max $\left\{\frac{-cu}{b}, \frac{cv}{a}\right\}$ be m. Then the solutions in positive integers correspond to $t = m + 1, m + 2, \ldots$ Clearly, the number of positive integral solutions is infinite.

Worked Examples (continued).

13. Find the general solution in integers of the equation $7x + 11y = 1$.

Since 7 and 11 are prime to each other, there exist integers u and v such that $7u + 11v = 1$. Here $u = 8, v = -5$.
Then $7x + 11y = 7.8 - 11.5$
or, $7(x - 8) = -11(y + 5)$.
Since 7 and 11 are prime to each other, $x - 8$ is divisible by 11 and $y + 5$ is divisible by 7 and therefore
$\frac{x-8}{-11} = \frac{y+5}{7} = t$, where t is an integer
or, $x = 8 - 11t$
$y = -5 + 7t$, where $t = 0, \pm 1, \pm 2, \cdots$

This is the general solution in integers.

Note. For a positive integral solution, we must have $8 - 11t > 0$ and $-5 + 7t > 0$ simultaneously. Hence $\frac{5}{7} < t < \frac{8}{11}$.
No such integer t exists. Hence there is no solution of the equation in positive integers.

14. Find the general solution in integers of the equation $5x + 12y = 80$. Examine if there is a solution in positive integers.

Since 5 and 12 are prime to each other, there exist integers u and v such that $5u + 12v = 1$. Here $u = 5, v = -2$.
Then $5x + 12y = 80(5.5 - 12.2)$
or, $5(x - 400) = -12(y + 160)$.
Since 5 and 12 are prime to each other, $x - 400$ is divisible by 12 and $y + 160$ is divisible by 5 and therefore
$\frac{x-400}{-12} = \frac{y+160}{5} = t$, where t is an integer
or, $x = 400 - 12t$
$y = 5t - 160$, where $t = 0, \pm 1, \pm 2, \cdots$

This is the general solution in integers.

For a positive integral solution, we must have $400 - 12t > 0$ and $5t - 160 > 0$ simultaneously. Hence $32 < t < \frac{100}{3}$.
The only solution in positive integers corresponds to $t = 33$ and the solution is $x = 4, y = 5$.

15. Find the general solution in positive integers of the equation $12x - 7y = 8$.

Since 12 and 7 are prime to each other, there exist integers u and v such that $12u + 7v = 1$. Here $u = 3, v = -5$.

Then $12x - 7y = 8(12.3 - 7.5)$

or, $12(x - 24) = 7(y - 40)$.

Since 12 and 7 are prime to each other, $x - 24$ is divisible by 7 and $y - 40$ is divisible by 12 and therefore

$\frac{x-24}{7} = \frac{y-40}{12} = t$, where t is an integer

or, $x = 7t + 24$

$y = 12t + 40$, where $t = 0, \pm 1, \pm 2, \cdots$

This is the general solution in integers.

For a solution in positive integers we must have $7t + 24 > 0$ and $12t + 40 > 0$. Hence $t > \frac{-24}{7}$ and $t > \frac{-10}{3}$.

The least integral value of t is -3. Hence the general solution in positive integers is given by $x = 7t + 24$

$y = 12t + 40$, where t is an integer ≥ -3.

Note. The solution corresponding to $t = -3$ is given by $x = 3, y = 4$. The general solution in positive integers can be expressed as

$x = 7t + 3 \quad y = 12t + 4$, where t is an integer ≥ 0.

3.3. Prime numbers.

An integer $p > 1$ is said to be a *prime number*, or simply a *prime*, if its only positive divisors are 1 and p.

An integer > 1 which is not a prime is said to be a *composite* number.

The integers $2, 3, 5, 7, 11, \ldots$ are prime numbers, while the integers $4, 6, 8, 9, \ldots$ are composite numbers.

The integer 1 is regarded as neither prime nor composite.

2 is the only even prime number. All other prime numbers are necessarily odd.

Theorem 3.3.1. If p be a prime number and $1 \leq a < p$ then p is prime to a.

Proof. Let $d = gcd(a, p)$. Then $d \mid a$ and $d \mid p$.

Since p is a prime and $d \mid p$, either $d = p$ or $d = 1$.

But since $a < p$ and $d \mid a$, d cannot be p. Therefore $d = 1$ and p is prime to a.

Theorem 3.3.2. If p be a prime number and a is an integer $> p$ such that p is not a divisor of a, then p is prime to a.

Proof. Let $d = gcd(a, p)$. Then $d \mid a$ and $d \mid p$.

Since p is a prime and $d \mid p$, either $d = p$ or $d = 1$.

But $d \neq p$ since p is not a divisor of a. Therefore $d = 1$ and p is prime to a.

Theorem 3.3.3. If p be a prime number and a is an integer $> p$ such that p is a divisor of a, then $gcd(a, p) = p$.

Proof. Since p is a divisor of a, $a = pk$ where k is an integer.

Hence $gcd(a, p) = gcd(pk, p) = p.gcd(k, 1) = p$.

Theorem 3.3.4. If p be a prime number and $p \mid ab$, then either $p \mid a$ or $p \mid b$.

Proof. If $p \mid a$ then the theorem is done.

If p is not a divisor of a then $gcd(a, p) = 1$, since 1 and p are the only divisors of p.

Since $gcd(a, p) = 1$, there exist integers u and v such that $au + pv = 1$.

Then $abu + pbv = b$.

Now $p \mid ab$ and $p \mid pb \Rightarrow p \mid (ab)u + (pb)v$, since u and v are integers.

That is, $p \mid b$.

This completes the proof.

Corollary. If p be a prime and $p \mid a_1 a_2 \ldots a_n$, then $p \mid a_k$ for some k where $1 \leq k \leq n$.

Proof. If $p \mid a_1$ we need not go further. If p is not a divisor of a_1 then by the theorem, $p \mid a_2 a_3 \ldots a_n$.

If p is not a divisor of a_2 then $p \mid a_3 a_4 \ldots a_n$. Proceeding in a similar manner, in a finite number of steps we arrive at the desired result.

Theorem 3.3.5. A composite number has at least one prime divisor.

Proof. Let n be a composite number. Since n is not a prime, it has a positive divisor other than 1 and n.

Let S be the set of those positive divisors of n which are different from 1 and n. Then S is non-empty. By the well ordering property of the set $\mathbb{N}$, S contains a least element, say d. Then $1 < d < n$.

We prove that d is a prime.

If d be not a prime then d has a divisor d' other than d and 1; and $1 < d' < d < n$. But $d' \mid d$ and $d \mid n \Rightarrow d' \mid n$. Therefore $d' \in S$ and this contradicts that d is the least element of S.

Therefore d is a prime and the theorem is done.

Worked Examples.

1. Prove that for $n > 3$, the integers n, $n+2$, $n+4$ cannot be all primes.

Any positive integer n is one of the forms $3k$, $3k+1$, $3k+2$, where k is a positive integer.

If $n = 3k$ then n is not a prime.

If $n = 3k+1$ then $n+2 = 3(k+1)$ and it is not a prime.

If $n = 3k+2$ then $n+4 = 3(k+2)$ and it is not a prime.

Thus in any case, the integers n, $n+2$, $n+4$ are not all primes.

2. p is a positive integer and $p, 2p+1, 4p+1$ are primes. Find p.

p is one of the forms $3k, 3k+1, 3k+2$, where k is an integer.

If $p = 3k+1$ then $2p+1 = 6k+3 = 3(2k+1)$ and it is not a prime.

If $p = 3k+2$ then $4p+1 = 12k+9 = 3(4k+3)$ and it is not a prime.

The only conclusion is $p = 3k$. Since p is a prime, $k = 1$. So $p = 3$.

3. If $p \geq q \geq 5$ and p, q are both primes, prove that $24 \mid (p^2 - q^2)$.

Since p and q are primes > 3, p and q are of the form $3k+1$ or $3k+2$, where k is an integer.

If both p and q are either of the forms $3k+1$ or $3k+2$, then $3 \mid (p-q)$.

If one of p and q is of the form $3k+1$ and the other is of the form $3k+2$, them $3 \mid (p+q)$.

Thus in any case, $3 \mid (p^2 - q^2)$.

Since p and q are odd primes, p and q are of the form $4k+1$ or $4k+3$, where k is an integer.

If both p and q are of the form $4k+1$, then $2 \mid (p+q)$ and $4 \mid (p-q)$.

If both p and q are of the form $4k+3$, then $2 \mid (p+q)$ and $4 \mid (p-q)$.

If one of p and q is of the form $4k+1$ and the other is of the form $4k+3$, then $4 \mid (p+q)$ and $2 \mid (p-q)$.

Thus in any case, $8 \mid (p^2 - q^2)$.

Since 3 and 8 are prime to each other, $24 \mid (p^2 - q^2)$.

4. If p and $p^2 + 8$ are both prime numbers, prove that $p = 3$.

Any integer p is one of the forms $3k$, $3k+1$, $3k+2$, where k is an integer.

If $p = 3k+1$, then $p^2 + 8 = 3(3k^2 + 2k + 3)$. Since $p^2 + 8$ is a prime, $3k^2 + 2k + 3$ must be 1 for some integer k and in that case $p^2 + 8$ must be 3.

But for no integer k, $3k^2 + 2k + 3$ can be 1 and for no integer k, $p^2 + 8$ can be 3. Therefore $p = 3k+1$ is an impossibility.

If $p = 3k+2$, then $p^2 + 8 = 3(3k^2 + 4k + 4)$. Since $p^2 + 8$ is a prime, $3k^2 + 4k + 4$ must be 1 for some integer k and in that case $p^2 + 8$ must be 3.

By similar arguments, $p = 3k+2$ is an impossibility.

Therefore $p = 3k$, where k is an integer. Since p is a prime, k must be 1 and therefore $p = 3$.

5. If $2^n - 1$ be a prime, prove that n is a prime.

Let n be composite. Then $n = p.q$ where p and q are integers each greater than 1.

$2^n - 1 = 2^{pq} - 1 = (2^p - 1)(2^{p(q-1)} + 2^{p(q-2)} + \cdots + 2^p + 1)$.

Each factor on the right is evidently greater than 1 and therefore $2^n - 1$ is composite.

Contrapositively, $2^n - 1$ is a prime implies n is a prime.

6. Prove that $n^4 + 4^n$ is a composite number for all $n > 1$.

Case 1. Let n be even.

Then $n^4 + 4^n$ is divisible by 4 and so it is a composite number.

Case 2. Let n be odd and $n = 2k+1$, where k is a natural number.

Then $n^4 + 4^n = n^4 + 4.4^{2k} = n^4 + 4a^4$, where $a = 2^k$

$= (n^2 + 2a^2)^2 - (2an)^2 = (n^2 + 2an + 2a^2)(n^2 - 2an + 2a^2)$.

$(n^2 + 2an + 2a^2) = (n+a)^2 + a^2$ and $(n^2 - 2an + 2a^2) = (n-a)^2 + a^2$.

Since a is a positive integer > 1, $(n+a)^2 + a^2 > 1$ and $(n-a)^2 + a^2 > 1$.

Consequently, $n^4 + 4^n$ is a composite number when n is odd.

Hence $n^4 + 4^n$ is a composite number for all $n > 1$.

7. Let p be a prime and a be a positive integer. Prove that a^n is divisible by p if and only if a is divisible by p.

Let a be divisible by p. Then $a = pk$ for some integer k.

$a^n = p^n k^n = p(p^{n-1} k^n) = pm$, where m is an integer.

This shows that a^n is divisible by p ... ... (i)

Let a be not divisible by p. Since p is a prime, $gcd(a, p) = 1$. Therefore there exist integers u and v such that $au + pv = 1$.

Then $a^n u^n = (1 - pv)^n = 1 - ps$ where s is an integer

or, $a^n r + ps = 1$ where r, s are integers.

This shows that $gcd(a^n, p) = 1$ and therefore a^n is not divisible by p.

Hence a is not divisible by $p \Rightarrow a^n$ is not divisible by p.

Contrapositively, $p \mid a^n \Rightarrow p \mid a$ ... ... (ii)

From (i) and (ii) the desired result is obtained.

Theorem 3.3.6. (Fundamental theorem of Arithmetic)

Any positive integer is either 1, or a prime, or it can be expressed as a product of primes, the representation being unique except for the order of the prime factors.

Proof. Let n be a positive integer. Either $n = 1$ or $n > 1$. Let $P(n)$ be the statement that $n(> 1)$ is either a prime, or it can be expressed as a product of primes.

$P(2)$ is true, since 2 is a prime.

Let us assume that $P(n)$ is true for all n, where n is a positive integer such that $2 \leq n \leq k$.

If $k+1$ be itself a prime then $P(k+1)$ is true and by the second principle of induction, $P(n)$ is true for all positive integers $n > 1$.

If $k+1$ be not a prime then it is a composite number. Let $k+1 = rs$ where r, s are integers with $2 \leq r < k+1, 2 \leq s < k+1$.

By induction hypothesis, $P(r)$ and $P(s)$ are both true. Then

$r = p_1 p_2 \ldots p_i$ where $p_1, p_2, \ldots, p_i$ are primes, $i \geq 1$;

$s = q_1 q_2 \ldots q_j$ where $q_1, q_2, \ldots, q_j$ are primes, $j \geq 1$.

Thus $k+1$ is expressed as the product of primes and $P(k+1)$ is proved to be true. By the second principle of induction $p(n)$ is true for all positive integers $n > 1$.

Hence the first part of the theorem is established.

In order to prove uniqueness of the representation, let us assume that $n = p_1 p_2 \ldots p_k = q_1 q_2 \ldots q_m$, where p_i and q_i are all primes.

Since $p_1 \mid n$, it follows that $p_1 \mid q_1 q_2 \ldots q_m$.

Since p_1 is a prime, $p_1 \mid q_r$ for some r where $1 \leq r \leq m$. But since p_1 and q_r are both primes, $p_1 = q_r$.

We obtain $p_2 p_3 \ldots p_k = q_1 q_2 \ldots q_{r-1} q_{r+1} \ldots q_m$.

We repeat the argument with p_2 and obtain $p_2 = q_s$ for some s, where $1 \leq s \leq m, s \neq r$. Then

$$p_3 p_4 \ldots p_k = q_1 q_2 \ldots q_{r-1} q_{r+1} \ldots q_{s-1} q_{s+1} \ldots q_m.$$

If $k < m$, then after k steps the left hand side reduces to 1 and the right hand side becomes the product of $m - k$ q's, each of which is a prime. This cannot happen. Therefore $k \geq m$.

If $k > m$, then after m steps the right hand side reduces to 1 and the left hand side becomes the product of $k - m$ p's, each of which is a prime. This cannot happen. Therefore $k \leq m$.

Hence $k = m$ and the products $p_1 p_2 \ldots p_m$ and $q_1 q_2 \ldots q_k$ give the same representation except for the order of the factors.

Thus $n(> 1)$ is expressed as the product of a number of primes, the representation being unique except for the order of the factors.

Note. In the application of the fundamental theorem we write any integer $n(> 1)$ in the form, called *the canonical form*,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r},$$

where the primes p_i are distinct with $p_1 < p_2 < \cdots < p_r$ and the exponents α_i are positive.

An integer is said to be *square-free* if no α_i in the canonical form of n is greater than 1.

To illustrate the representation, let us take $n = 3150, 210$.

$3150 = 2.3.3.5.5.7 = 2.3^2.5^2.7$. $210 = 2.3.5.7$. 210 is square free.

Another representation. Every positive integer n (including 1) can also be expressed as $n = \prod_p p^{\alpha(p)}$, where p is a prime and $\alpha(p)$ is an integer ≥ 0 and $\alpha(p) = 0$ for sufficiently large p.

In particular, if $n = 1$, then each $\alpha(p) = 0$ and in this case the product $\prod_p p^{\alpha(p)}$ becomes empty.

The symbol $\alpha(p)$ indicates that α depends on the prime p.

If $a = \prod_p p^{\alpha(p)}, b = \prod_p p^{\beta(p)}$ be two positive integers then

the g.c.d. of a and b is $\prod_p p^{\gamma(p)}$, where $\gamma(p) = \min\{\alpha(p), \beta(p)\}$ and

the l.c.m. of a and b is $\prod_p p^{\delta(p)}$, where $\delta(p) = \max\{\alpha(p), \beta(p)\}$.

Since $\max\{\alpha, \beta\} + \min\{\alpha, \beta\} = \alpha + \beta$ for any two real numbers α and β, it follows that for any two positive integers a, b, $(a, b).[a, b] = a.b$.

For example, let $a = 60, b = 63, c = 1$. Then

$a = 2^2.3.5.7^0$, $b = 2^0.3^2.5^0.7$, $c = 2^0.3^0.5^0.7^0$.

$(a, b) = 2^0.3.5^0.7^0 = 3$, $(a, c) = 2^0.3^0.5^0.7^0 = 1$,

$[a, b] = 2^2.3^2.5.7 = 1260$, $[a, c] = 2^2.3.5.7^0 = 60$.

$(a, b).[a, b] = 3.1260 = 3780 = a.b$; $(a, c).[a, c] = 1.60 = a.c$.

Theorem 3.3.7. If a, b, c be positive integers then

(i) $(a, [b, c]) = [(a, b), (a, c)]$;

(ii) $[a, (b, c)] = ([a, b], [a, c])$.

To prove the theorem we first prove the following lemma.

Lemma. If α, β, γ be integers then

(i) $\min\{\alpha, \max\{\beta,\gamma\}\} = \max\{\min\{\alpha,\beta\}, \min\{\alpha,\gamma\}\}$

(ii) $\max\{\alpha, \min\{\beta,\gamma\}\} = \min\{\max\{\alpha,\beta\}, \max\{\alpha,\gamma\}\}$.

Proof. (i) Let us consider the following cases.

Case 1. $\alpha \le \min\{\beta,\gamma\}$. Then $\alpha \le \max\{\beta,\gamma\}$.
L.H.S. $= \alpha$, R.H.S.$= \max\{\alpha,\alpha\} = \alpha$.

Case 2. $\alpha \ge \max\{\beta,\gamma\}$. Then $\alpha \ge \beta, \alpha \ge \gamma$.
L.H.S. $= \max\{\beta,\gamma\}$, R.H.S.$= \max\{\beta,\gamma\}$.

Case 3. $\beta \le \alpha \le \gamma$.
L.H.S. $= \min\{\alpha,\gamma\} = \alpha$, R.H.S.$= \max\{\beta,\alpha\} = \alpha$.

Case 4. $\gamma \le \alpha \le \beta$.
L.H.S. $= \min\{\alpha,\beta\} = \alpha$, R.H.S. $= \max\{\alpha,\gamma\} = \alpha$.

This completes the proof.

(ii) Similar proof.

Proof of the theorem.

In consequence of the fundamental theorem of arithmetic, there exis prime numbers $p_1, p_2, \ldots, p_n$ such that

$a = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, where α_i are integers ≥ 0,

$b = p_1^{\beta_1} p_2^{\beta_2} \ldots p_n^{\beta_n}$, where β_i are integers ≥ 0,

$c = p_1^{\gamma_1} p_2^{\gamma_2} \ldots p_n^{\gamma_n}$, where γ_i are integers ≥ 0.

$[b,c] = p_1^{x_1} p_2^{x_2} \ldots p_n^{x_n}$, where $x_i = \max\{\beta_i, \gamma_i\}$, $i = 1, 2, \ldots, n$.

$(a,[b,c]) = p_1^{y_1} p_2^{y_2} \ldots p_n^{y_n}$, where $y_i = \min\{\alpha_i, x_i\}$, $i = 1, 2, \ldots, n$.

$(a,b) = p_1^{w_1} p_2^{w_2} \ldots p_n^{w_n}$, where $w_i = \min\{\alpha_i, \beta_i\}$, $i = 1, 2, \ldots, n$.

$(a,c) = p_1^{z_1} p_2^{z_2} \ldots p_n^{z_n}$, where $z_i = \min\{\alpha_i, \gamma_i\}$, $i = 1, 2, \ldots, n$.

$[(a,b),(a,c)] = p_1^{t_1} p_2^{t_2} \ldots p_n^{t_n}$, where $t_i = \max\{w_i, z_i\}$, $i = 1, 2, \ldots, n$.

By the lemma, we have $y_i = \min\{\alpha_i, x_i\}$

$= \min\{\alpha_i, \max\{\beta_i, \gamma_i\}\}$

$= \max\{\min\{\alpha_i, \beta_i\}, \min\{\alpha_i, \gamma_i\}\}$

$= \max\{w_i, z_i\} = t_i, i = 1, 2, \ldots n.$

Therefore $(a,[b,c]) = [(a,b),(a,c)]$.

(ii) Similar proof by using lemma (ii).

Note. If we define binary operations $\circ$ and $\star$ on the set $\mathbb{N}$ by

$a \circ b =$ the g.c.d. of a and b,

$a \star b =$ the l.c.m. of a and b for $a, b \in \mathbb{N}$

then the part (i) of the theorem states that $a \circ (b \star c) = (a \circ b) \star (a \circ c)$ for $a, b \in \mathbb{N}$ and the part (ii) of the theorem states that $a \star (b \circ c) = (a \star b) \circ (a \star c)$ for $a, b \in \mathbb{N}$.

These establish that the operation $\circ$ is distributive over the operation $\star$ and the operation $\star$ is distributive over the operation $\circ$.

Worked Examples.

1. If $2^n + 1$ is an odd prime for some integer n, prove that n is a power of 2.

If n is odd, $2^n + 1$ is divisible by $2 + 1$ and therefore $2^n + 1$ is not a prime. So n is even.

Let $n = 2^k p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where p_i's are primes.

Then $n = 2^k . p$, where p is an odd integer.

If $p > 1$, $2^n + 1 = 2^{2^k . p} + 1 = (2^{2^k})^p + 1$ and it is divisible by $2^{2^k} + 1$, since p is odd. This contradicts that $2^n + 1$ is a prime.

Consequently, $p = 1$ and $n = 2^k$.

2. If p be a prime, show that $\sqrt{p}$ is not a rational number.

Since p is a prime, p is an integer ≥ 2 and therefore $\sqrt{p} > 1$.

Let $\sqrt{p}$ be a rational number. Then $\sqrt{p} = \frac{m}{n}$ for some natural numbers m, n. We assert that $m > 1$ and $n > 1$, because

$m = 1$ and $n = 1 \Rightarrow p = 1^2 = 1$, a contradiction

$m > 1$ and $n = 1 \Rightarrow p = m.m$ and therefore p is not a prime, a contradiction

$m = 1$ and $n > 1 \Rightarrow \sqrt{p} < 1$, a contradiction.

Therefore $m > 1$ and $n > 1$. We also have $pn^2 = m^2$. The number of primes in the factorisation of m being unique by the fundamental theorem of arithmetic, it follows that the number of primes (counting multiplicity) in the factorisation of m^2 is always even.

Similarly, the number of primes in the factorisation of n^2 is also even. Therefore the number of primes in the factorisation of pn^2 is odd (since p is a prime).

Since $pn^2 = m^2$, it appears that the same integer m^2 is expressed as the product of an odd number of primes in one representation and as the product of an even number of primes in another representation.

This contradicts uniqueness of the number of prime factors in the decompostion. We conclude that $\sqrt{p}$ is not a rational number.

Theorem 3.3.8. (Euclid) The number of primes is infinite.

Proof. We prove the theorem by contradiction.

Let us suppose that the number of primes is finite and let p be the greatest prime. We write the primes $2, 3, 5, 7, \ldots$ in succession and p is the last in the enumeration.

The product $2.3.5 \ldots p$ in which every prime appears only once is divisible by each prime and therefore the number $(2.3.5 \ldots p) + 1$ is not divisible by any of the primes $2, 3, 5, \ldots, p$.

Hence this number is either itself a prime, or being a composite number, is divisible by a prime number greater than p. In both the cases p fails to be the greatest prime and therefore the number of primes is infinite.

Note. Although the number of primes is infinite, there are arbitrarily large gaps in the sequence of primes. For every positive integer k, there exist k consecutive composite numbers. To be explicit, each of the k consecutive integers

$$(k+1)! + 2, (k+1)! + 3, \ldots, (k+1)! + (k+1)$$

is composite, because $(k+1)! + r$ is divisible by r if $2 \leq r \leq k+1$.

This indicates that the primes are irregularly spaced in the sequence of positive integers. The number of primes less than a positive integer x is denoted by $\pi(x)$. No simple formula for determining $\pi(x)$ has yet been found.

Test for primality.

If a positive integer a be composite, then $a = bc$ for integers b, c satisfying $1 < b < a$, $1 < c < a$. Let $b \leq c$. Then $b^2 \leq bc = a$ and this implies $b \leq \sqrt{a}$.

Since $b > 1$, b has at least one prime divisor p and $p \leq b \leq \sqrt{a}$.

In testing primality of a positive integer n, it is sufficient to divide n by primes not exceeding $\sqrt{n}$.

Greek mathematician, Eratosthenes (276 - 494 B.C.) utilised this concept to find all primes less than a given positive integer n. His device is called the "seive of Eratosthenes" which consists in writing all integers from 2 to n in natural order and then striking out all multiples $2p, 3p, 4p, 5p, \ldots$ of all primes $p \leq \sqrt{n}$. The integers that are left in the list (survived the seive) are primes.

For example, in order to determine all primes ≤ 30, the "sieve" method is applied by striking all multiples of $2, 3, 5$ from the table of integers from 2 to 30, since 5 is the largest prime $\leq \sqrt{30}$.

The table is shown below.

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~ 13 ~~14~~ ~~15~~

~~16~~ 17 ~~18~~ 19 ~~20~~ 21 ~~22~~ 23 ~~24~~ ~~25~~ ~~26~~ ~~27~~ ~~28~~ 29 ~~30~~

This method has limitations. If the positive integer n be sufficiently large, the method becomes impracticable.

Distribution of primes.

According to Divison algorithm, all odd positive integers fall into two progressions – one containing positive integers of the form $4n + 1$ and the other containing positive integers of the form $4n + 3$, where $n \in \mathbb{N}$.

It is natural to ask whether the infinitude of primes ultimately fall into one particular progression or they are infinitely distributed in both the progressions.

In this respect a more general theorem has been established by Dirichlet which states that – if a and b are positive integers relatively prime then the arithmetic progression $a + nb$ ($n \in \mathbb{N}$) contains infinitely many primes.

The proof of the general theorem requires analytic method and as such it is beyond the scope of the book. In special cases the proofs can be carried along the same line as in Euclid's theorem. We consider one such.

Theorem 3.3.9. (Dirichlet). There are infinitely many primes of the form $4n - 1$.

Proof. We assume that there are only a finite number of primes of the form $4n - 1$. Let p be the largest prime of that form.

Let us consider the positive integer $N = (2^2.3.5...p) - 1$. The product $3.5...p$ contains all odd primes $\leq p$ as divisors.

Since N is of the form $4n - 1$ and $N > p$, N cannot be a prime, because by assumption, p is the largest prime of the form $4n - 1$.

As N is composite, N has prime divisors and they are all greater than p. All of Them cannot be of the form $4n + 1$, because the product of two numbers of the form $4n + 1$ is of the same form.

Therefore some prime divisor of N must be of the form $4n - 1$ and clearly that prime number is greater than p and this contradicts our assumption that p is the largest prime of the form $4n - 1$.

The contradiction proves the theorem.

3.3.10. The number of positive divisors of a positive integer.

Let n be a positive integer greater than 1. Then n can be expressed as $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where the primes p_i are distinct with $p_1 < \dots < p_r$ and the exponents α_i are all positive.

If m be a positive divisor of n then m is of the form $p_1^{u_1} p_2^{u_2} \dots p_r^{u_r}$, where $0 \le u_1 \le \alpha_1, 0 \le u_2 \le \alpha_2, \dots, 0 \le u_r \le \alpha_r$.

Thus the positive divisors of n are in one-to-one correspondence with the totality of r-tuples $(u_1, u_2, \dots, u_r)$, where $0 \le u_1 \le \alpha_1, 0 \le u_2 \le \alpha_2, \dots, 0 \le u_r \le \alpha_r$.

The number of such r-tuples is $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)$.

Hence the total number of positive divisors of n is $(\alpha_1+1)(\alpha_2+1) \cdots (\alpha_r+1)$.

If $n = 1$, then there is only one positive divisor.

Note. The total number of positive divisors $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)$ include both the divisors 1 and n.

Definition. The number of positive divisors of a positive integer n is denoted by $\tau(n)$. (*tau n*)

If the canonical form of a positive integer $n(> 1)$ be

$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $p_1, p_2, \dots, p_r$ and distinct primes and the exponents α_i are all positive,

then $\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)$; and $\tau(1) = 1$.

For example, $\tau(48) = \tau(2^4.3) = (4 + 1)(1 + 1) = 10$.

Theorem 3.3.11. The total number of positive divisors of a positive integer n is odd if and only if n is a perfect square.

Proof. Let $n(> 1)$ be a perfect square and let the canonical form of n be $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $p_1 < p_2 < \dots < p_r$ and α_i are all positive.

Then each of $\alpha_1, \alpha_2, \dots, \alpha_r$ is an even integer and $\tau(n) = (\alpha_1+1)(\alpha_2+1) \cdots (\alpha_r + 1)$ is odd.

If however, $n = 1$, a perfect square, then $\tau(n) = 1$ and it is odd.

Conversely, let $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)$ be odd. Then each of the factors $\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_r + 1$ must be odd. Consequently, each of $\alpha_1, \alpha_2, \dots, \alpha_r$ must be even and n is therefore a perfect square.

This completes the proof.

Worked Examples.

1. Find $\tau(360)$ and $\tau(900)$.

$360 = 2^3.3^2.5$. Therefore $\tau(360) = (1 + 3).(1 + 2).(1 + 1) = 24$.

$900 = 2^2.3^2.5^2$. Therefore $\tau(900) = (1 + 2).(1 + 2).(1 + 2) = 27$.

2. Find the number of odd positive divisors of 2700.

$2700 = 2^2.3^3.5^2$. Every positive divisor of 2700 is of the form $2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}$, where $0 \le \alpha_1 \le 2, 0 \le \alpha_2 \le 3, 0 \le \alpha_3 \le 2$.

Each term in the product $(1 + 2 + 2^2)(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2)$ is a positive divisor of 2700 and conversely.

The odd positive divisors of 2700 are given by the terms of the product $1.(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2)$.

The number of odd positive divisors are $(3 + 1)(2 + 1)$, i.e., 12.

3. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots, p_k$ are primes and $\alpha_i \ge 1$, prove that the number of positive square-free divisors of n is 2^k.

A positive square-free divisor of n is of the form $p_1^{u_1} p_2^{u_2} \dots p_k^{u_k}$, where $0 \le u_1 \le 1, 0 \le u_2 \le 1, \dots, 0 \le u_k \le 1$ and they are in one-to-one correspondence with the totality of r-tuples $(u_1, u_2, \dots, u_r)$, where $0 \le u_1 \le 1, 0 \le u_2 \le 1, \dots, 0 \le u_k \le 1$.

The number of such r-tuples is 2^k.

Hence the number of positive square-free divisors of n is 2^k.

4. Find the smallest number having 8 positive divisors.

$8 = 2.2.2 = 2^2.2 = 2^3$.

Let n be a number with 8 positive divisors.

The factorisation of 8 as $8 = 2.2.2$ indicates that the number n is of the form $p_1.p_2.p_3$, where p_1, p_2, p_3 are distinct primes. For example, $n = 2.3.7$.

The factorisation of 8 as $8 = 4.2$ indicates that number n is of the form $p_1^3.p_2$, where p_1, p_2 are distinct primes. For example, $n = 3^3.2$.

The factorisation of 8 as $8 = 8.1$ indicates that the number n is of the form p_1^7, where p_1 is a prime. For example, $n = 2^7$.

Therefore the number n is of one of the forms $p_1.p_2.p_3$, $p_1^3.p_2$, p_1^7, where p_1, p_2, p_3 are distinct primes.

Clearly, the least number with 8 positive divisors is $2^3.3 = 24$.

3.3.12. The sum of all positive divisors of a positive integer.

Let n be a positive integer greater than 1. Then n can be expressed as $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where the primes p_i are distinct with $p_1 < p_2 < \dots < p_r$ and $\alpha_i > 0$.

Every positive divisor of n is a term in the product

$$(1 + p_1 + p_1^2 + \dots + p_1^{\alpha_1})(1 + p_2 + \dots + p_2^{\alpha_2}) \dots (1 + p_r + \dots + p_r^{\alpha_r})$$

and conversely, each term in the product is a divisor of n.

Hence the sum of all positive divisors of n

$$= (1+p_1+p_1^2+\dots+p_1^{\alpha_1})(1+p_2+p_2^2+\dots+p_2^{\alpha_2}) \dots (1+p_r+p_r^2+\dots+p_r^{\alpha_r})$$

$$= \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdot \dots \cdot \frac{p_r^{\alpha_r+1}-1}{p_r-1}.$$

If $n = 1$, 1 is the only divisor of n and the sum $=1$.

Definition. The sum of all positive divisors of a positive integer n is denoted by $\sigma(n)$. (*sigma n*).

If the canonical form of a positive integer $n(> 1)$ be

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r},$$

then $\sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdot \dots \cdot \frac{p_r^{\alpha_r+1}-1}{p_r-1}$; and $\sigma(1) = 1$.

Definition. A function whose domain is the set of all positive integers is said to be a *number-theoretic function* (or an *arithmetic function*).

The range of a number-theoretic function need not be the set of all positive integers. We shall encounter some simple number-theoretic functions which assume positive integral values.

The functions τ and σ are examples of number-theoretic functions.

A number-theoretic function f is said to be *multiplicative* if $f(mn) = f(m)f(n)$ for all integers m, n such that m, n are prime to each other.

Theorem 3.3.13. The functions τ and σ are both multiplicative functions.

Proof. Let m, n be relatively prime integers.

$\tau(mn) = \tau(m)\tau(n)$ holds trivially if either m is 1 or n is 1.

We assume $m > 1$ and $n > 1$.

Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ and $n = q_1^{\beta_1} q_2^{\beta_2} \dots q_s^{\beta_s}$, where p_i, q_j are primes and $\alpha_i \geq 1$, $\beta_j \geq 1$.

Since m, n are relatively prime, each p_i is different from each q_j.

Therefore the prime factorisation of mn is

$mn = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r} q_1^{\beta_1} q_2^{\beta_2} \dots q_s^{\beta_s}$.

$$\tau(mn) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_r + 1)(\beta_1 + 1)(\beta_2 + 1) \dots (\beta_s + 1)$$
$$= \tau(m)\tau(n).$$

$$\sigma(mn) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \dots \frac{p_r^{\alpha_r+1}-1}{p_r-1} \cdot \frac{q_1^{\beta_1+1}-1}{q_1-1} \cdot \frac{q_2^{\beta_2+1}-1}{q_2-1} \dots \frac{q_s^{\beta_s+1}-1}{q_s-1};$$
$$= \sigma(m)\sigma(n).$$

Hence τ and σ are multiplicative functions.

Definition. Perfect number. A positive integer n is said to be a *perfect number* if $\sigma(n) = 2n$, i.e., if n be the sum of all its positive divisors excluding itself.

For example, 6 is a perfect number; 28 is another.

Worked Examples.

1. Find $\sigma(360)$ and $\sigma(900)$.

$360 = 2^3.3^2.5$. Therefore $\sigma(360) = \frac{2^4-1}{2-1} \cdot \frac{3^3-1}{3-1} \cdot \frac{5^2-1}{5-1} = 15.13.6 = 1170$.

$900 = 2^2.3^2.5^2$. Therefore $\sigma(900) = \frac{2^3-1}{2-1} \cdot \frac{3^3-1}{3-1} \cdot \frac{5^3-1}{5-1} = 7.13.31 = 2821$.

2. Find the sum of all even positive divisors of 2700.

$2700 = 2^2.3^3.5^2$. Every positive divisor of 2700 is of the form $2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}$, where $0 \leq \alpha_1 \leq 2, 0 \leq \alpha_2 \leq 3, 0 \leq \alpha_3 \leq 2$.

Therefore each term in the product $(1+2+2^2)(1+3+3^2+3^3)(1+5+5^2)$ is a positive divisor of 2700 and conversely.

The even positive divisors of 2700 are given by the different terms of the product

$$(2 + 2^2)(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2).$$

The sum of the even positive divisors

$$= (2 + 2^2)(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2) = 6.40.31 = 7440.$$

3. Let $k > 1$ and $2^k - 1$ is a prime. If $n = 2^{k-1}(2^k - 1)$ then show that n is a perfect number.

$2^k - 1$ is an odd prime, say p.

$\sigma(n) = \sigma(2^{k-1}p) = \sigma(2^{k-1})\sigma(p)$, since 2^{k-1} and p are prime to each other.

$\sigma(2^{k-1}) = 1 + 2 + 2^2 + \dots + 2^{k-1} = 2^k - 1$ and $\sigma(p) = 1 + p$.

Therefore $\sigma(n) = (2^k - 1)(1 + p) = (2^k - 1)2^k = 2n$.

This proves that n is a perfect number.

Note. This example shows that if $2^n - 1$ $(n > 1)$ is a prime, then the number $2^{n-1}(2^n - 1)$ is a perfect number.

The numbers of the form $M_n = 2^n - 1$ $(n > 1)$ are called Mersenne numbers, named after Mersenne (1588-1648), a French monk and an amateur of mathematics.

The primality of M_n requires n must be a prime.

If M_n be a prime then M_n is called a Mersenne prime and in that case a perfect number $2^{n-1}(2^n - 1)$ is obtained.

4. If $d_1, d_2, \ldots, d_k$ be the list of all positive divisors of a positive integer n, prove that $\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = \frac{\sigma(n)}{n}$.

d_i is a positive divisor $\Rightarrow \frac{n}{d_i}$ is also a positive divisor. As d runs through the set of all positive divisors of n, $\frac{n}{d}$ also does so.

Therefore $\frac{n}{d_1} + \frac{n}{d_2} + \cdots + \frac{n}{d_k} = d_1 + d_2 + \cdots + d_k = \sigma(n)$

or, $\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = \frac{\sigma(n)}{n}$.

Exercises 3A

1. Use the principle of induction to prove that
   (i) $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all $n \in \mathbb{N}$,
   (ii) $1.1! + 2.2! + \cdots + n.n! = (n+1)! - 1$ for all $n \in \mathbb{N}$,
   (iii) $3^{2n-1} + 2^{n+1}$ is divisible by 7 for all $n \in \mathbb{N}$,
   (iv) $3^{4n+2} + 5^{2n+1}$ is divisible by 14 for all $n \in \mathbb{N}$,
   (v) $10^{n+1} + 10^n + 1$ is divisible by 3 for all $n \in \mathbb{N}$,
   (vi) $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n \in \mathbb{N}$.

2. Use the second principle of induction to prove that
   (i) $(3+\sqrt{7})^n + (3-\sqrt{7})^n$ is an even integer for all $n \in \mathbb{N}$,
   (ii) $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is divisible by 2^n for all $n \in \mathbb{N}$.

3. Prove that
   (i) the square of any integer is of the form $5k$ or $5k \pm 1$,
   (ii) the square of any integer is of the form $3k$ or $3k+1$,
   (iii) the cube of any integer is of the form $9k$ or $9k \pm 1$.

4. (i) If a be any integer, prove that $6 \mid a(a+1)(2a+1)$.
   (ii) If a be an odd integer, prove that $24 \mid a(a^2-1)$.
   (iii) If a be an odd integer, prove that $32 \mid (a^2+3)(a^2+7)$.
   (iv) If a be an even integer, prove that $48 \mid a(a^2+20)$.
   (v) If p be a prime number > 3, prove that $24 \mid p^2 - 1$.

5. Prove that
   (i) $1^n - 3^n - 6^n + 8^n$ is divisible by 10 for all $n \in \mathbb{N}$,
   (ii) $2^n - 5^n - 6^n + 9^n$ is divisible by 12 for all $n \in \mathbb{N}$.

[Hint. (i) $(1^n - 3^n) + (8^n - 6^n)$ is divisible by 2 and $(1^n - 6^n) + (8^n - 3^n)$ is divisible by 5.]

6. If $gcd(a,b) = au + bv$, where u and v are integers, prove that $gcd(u,v) = 1$.

[Hint. Let $d = gcd(a,b)$. $a = dp, b = dq$ for some integers p, q. $d = au + bv$ gives $pu + qv = 1$.]

7. (i) If a be an integer, prove that for all positive integers n, $gcd(a, a+n)$ is a divisor of n. Deduce that $gcd(a, a+1) = 1$.
   (ii) Show that for all odd inegers n, $gcd(3n, 3n+2) = 1$.
   (iii) If $gcd(a,b) = 1$, show that $gcd(a^2 - b^2, a^2 + b^2) = 1$ or 2.
   (iv) Prove that $gcd(2^{2^m} + 1, 2^{2^n} + 1) = 1$, if m, n are positive integers.

[Hint. (iv) Let $m > n$. Then $2^m = 2^n.2k$, for some integer k. Let $2^{2^n} = x$. Then $2^{2^m} - 1 = x^{2k} - 1$ and it is divisible by $x + 1$.]

8. (i) If a is prime to b and c is a divisor of a, prove that c is prime to b.
   (ii) If a is prime to b, prove that $a^2 + b^2$ is prime to $a^2 b^2$.

9. Use Euclidean algorithm to find integers u and v such that
   (i) $gcd(72, 120) = 72u + 120v$ (ii) $gcd(13, 80) = 13u + 80v$.

10. Find integers u and v satisfying
    (i) $20u + 63v = 1$, (ii) $30u + 72v = 12$, (iii) $52u - 91v = 78$.

11. Show that fraction is irreducible for all $n \in \mathbb{N}$.
    (i) $\frac{9n+8}{6n+5}$, (ii) $\frac{8n+5}{5n+3}$, (iii) $\frac{15n+4}{12n+3}$.

[Hint. (i) Let $a = 9n + 8$, $b = 6n + 5$. Then $2a - 3b = 1$. Therefore a and b are relatively prime.]

12. Find the general solution in integers and the least positive integral solutions of the equation
    (i) $8x - 27y = 1$, (ii) $12x - 17y = -1$,
    (iii) $35x - 13y = 10$, (iv) $41x - 17y = 8$,
    (v) $29x - 13y = 5$, (vi) $63x - 55y = -1$.

13. Solve the equation in positive integers
    (i) $11x + 7y = 151$, (ii) $13x + 4y = 115$,
    (iii) $9x + 25y = 311$, (iv) $12x + 5y = 149$.

14. (i) The sum of two positive integers is 200. If one is divided by 5 and the other is divided by 9, the remainder is 1 in each case. Find the numbers.

(ii) The sum of two positive integers is 100. If one is divided by 5, the remainder is 3 and if the other is divided by 9, the remainder is 4. Find the numbers.

15. (i) Prove that for all integers $n > 1$, $n^4 + 4$ is composite.

(ii) Prove that for all integers $n > 2$, $n^3 - 1$ is composite.

(iii) Prove that if n be composite, $2^n - 1$ is composite.

16. (i) Prove that the sum of first n natural numbers ($n > 2$) cannot be prime.

(ii) The only prime p such that $p+1$ is a perfect square is 3.

(iii) The only prime of the form $n^2 - 4$, n being an integer, is 3.

17. Prove that the sum of a pair of twin primes, each greater than 3, is divisible by 12.

[A pair of successive odd integers both of which are primes is said to be a *twin prime*.]

[**Hint.** Let $p, p+2$ be twin primes. Then $p+1$ is even and is divisible by 3.]

18. Find $\tau(n)$, where

(i) $n = 144$, (ii) $n = 450$, (iii) $n = 1482$, (iv) $n = 1932$.

19. Find $\sigma(n)$, where

(i) $n = 99$, (ii) $n = 210$, (iii) $n = 1225$ (iv) $n = 7007$.

20. Find the sum of all odd positive divisors and all even positive divisors of (i) 3600, (ii) 6300.

21. Find the least positive integer with 24 positive divisors.

22. Prove that the product of all positive divisors of a positive integer $n > 1$ is $n^{\tau(n)/2}$.

[**Hint.** Let d be a positive divisor of n. Then $dd' = n$ for some positive divisor d' of n. As d runs through all $\tau(n)$ positive divisors of n, d' does so. Therefore $\Pi d.\Pi d' = n^{\tau(n)}$. But $\Pi d = \Pi d'$.]

23. If n be a perfect number and $d_1, d_2, \ldots, d_k$ be the list of all positive divisors of n, prove that

$\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = 2$.

[**Hint.** If n be a perfect number, $\sigma(n) = 2n$.]

24. Let $k > 1$ and $2^k - 3$ is a prime. If $n = 2^{k-1}(2^k - 3)$ then show that $\sigma(n) = 2n + 2$.

3.4. Congruence.

Karl Friedrich Gauss (1777-1855), a celebrated German mathematician, introduced the concept of congruence which laid the foundation of modern theory of numbers.

Definition. Let m be a fixed positive integer. Two integers a and b are said to be *congruent modulo* m if $a - b$ is divisible by m. symbolically this is expressed as $a \equiv b \pmod{m}$.

To illustrate, let $m = 3$. It is easy to verify that

$1 \equiv 4 \pmod 3$, $-2 \equiv 1 \pmod 3$, $6 \equiv 0 \pmod 3$, $35 \equiv 2 \pmod 3$.

When $a - b$ is not divisible by m, a is said to be *incongruent* to b modulo m. It is expressed as $a \not\equiv b \pmod m$.

For example, $1 \not\equiv 5 \pmod 3$, $-2 \not\equiv 2 \pmod 3$.

Note. When $m = 1$, every two integers are congruent modulo m and this case is not so useful and interesting. Therefore m is usually taken to be a positive integer greater than 1.

Theorem 3.4.1. For any two integers a and b, $a \equiv b \pmod m$ if and only if a and b leave the same remainder when divided by m.

Proof. Let r be the remainder when a is divided by m. Then there exists some integer q such that $a = qm + r, 0 \le r < m$.

Since $a \equiv b \pmod m$, $a - b = km$ where k is an integer.

$$\begin{aligned} \text{Therefore } b = a - km &= (qm + r) - km \\ &= (q - k)m + r \end{aligned}$$

and this shows that b leaves the same remainder r.

Conversely, let r be the same remainder when a and b are divided by m. Then $a = q_1 m + r, b = q_2 m + r$, where q_1, q_2 are integers and $0 \le r < m$.

Therefore $(a - b) = (q_1 - q_2)m$, i.e., $m \mid a - b$ and this proves that $a \equiv b \pmod m$.

To illustrate, let $m = 5$. Since $21 \equiv 4.5 + 1$ and $-14 = -3.5 + 1$, 21 and -14 leave the same remainder upon division by 5. Therefore $21 \equiv -14 \pmod 5$.

Properties.

1. $a \equiv a \pmod m$.
2. If $a \equiv b \pmod m$ then $b \equiv a \pmod m$.
3. If $a \equiv b \pmod m, b \equiv c \pmod m$ then $a \equiv c \pmod m$.

4. If $a \equiv b \pmod m$ then for any integer c
$$a+c \equiv b+c \pmod m$$
$$ac \equiv bc \pmod m.$$

5. If $a \equiv b \pmod m$ and $c \equiv d \pmod m$ then
$$a+c \equiv b+d \pmod m$$
$$ac \equiv bd \pmod m.$$

6. If $a \equiv b \pmod m$ and $d|m, d>0$, then $a \equiv b \pmod d$.

Proofs of properties $1-4$ and 6 are immediate.

Proof. 5. $a \equiv b \pmod m \Rightarrow a-b = km$ and
$c \equiv d \pmod m \Rightarrow c-d = lm$, where k, l are integers.
$(a+c)-(b+d) = (k+l)m$.

Therefore $a+c \equiv b+d \pmod m$ since $k+l$ is an integer.

By property 4,
$a \equiv b \pmod m \Rightarrow ac \equiv bc \pmod m$ and
$c \equiv d \pmod m \Rightarrow bc \equiv bd \pmod m$.

Therefore $a \equiv b \pmod m$ and $c \equiv d \pmod m \Rightarrow ac \equiv bd \pmod m$.

Definition. If $a \equiv b \pmod m$ then b is said to be a *residue* of a modulo m.

By division algorithm there exist integers q and r satisfying $a = qm+r$ with $0 \le r \le m-1$.

Since $a-r = qm, a \equiv r \pmod m$ and this shows that r is a residue of a modulo m. r is said to be the *least non-negative residue* of a modulo m.

Let a be an arbitrary integer. Upon division by m, a leaves one and only one of the integers $0, 1, 2, \ldots, m-1$ as the remainder.

Therefore whatever the integer a may be, the least non-negative residue of a is one and only one of $0, 1, 2, \ldots, m-1$.

The whole set of integers is divided into m distinct and disjoint subsets, called the *residue classes modulo* m, denoted by $\bar{0}, \bar{1}, \bar{2}, \cdots, \overline{m-1}$ and defined by

$\bar{0} = \{0, \pm m, \pm 2m, \ldots\}$

$\bar{1} = \{1, 1 \pm m, 1 \pm 2m, \ldots\}$

$\bar{2} = \{2, 2 \pm m, 2 \pm 2m, \ldots\}$

$\cdots \quad \cdots \quad \cdots$

$\overline{m-1} = \{m-1, (m-1) \pm m, (m-1) \pm 2m, \ldots\}$.

Any two integers in a residue class are congruent modulo m and any two integers belonging to two different residue classes are incongruent modulo m.

Theorem 3.4.2. If $a \equiv b \pmod m$ then $a^n \equiv b^n \pmod m$ for all positive integers n.

Proof. We use the principle of induction to prove the theorem.

The theorem is true for $n = 1$.

Let us assume that the theorem is true for some positive integer k.
Then $a^k \equiv b^k \pmod m$.

Now $a^k \equiv b^k \pmod m$ and $a \equiv b \pmod m$ together imply that
$a^k.a \equiv b^k.b \pmod m$, i.e., $a^{k+1} \equiv b^{k+1} \pmod m$.

This shows that the thoerem is true for the positive integer $k+1$ if we assume it to be true for k.

By the principle of induction, the theorem is true for all positive integers n.

Note. The converse of the theorem fails to hold.
$a^k \equiv b^k \pmod m$ does not necessarily imply $a \equiv b \pmod m$.

For example, $9^2 \equiv 7^2 \pmod 8$ but $9 \not\equiv 7 \pmod 8$
$4^3 \equiv 7^3 \pmod 9$ but $4 \not\equiv 7 \pmod 9$.

Theorem 3.4.3. If $ax \equiv ay \pmod m$ and a is prime to m then $x \equiv y \pmod m$.

Proof. $ax - ay = km$, where k is an integer
or, $x - y = \frac{km}{a}$.

Since $x-y$ is an integer, $a \mid km$. Since a is prime to m and $a \mid km$, it follows that $a \mid k$. Therefore $k = aq$ where q is an integer.

Hence $x - y = qm$ and this proves the theorem.

Note. $ax \equiv ay \pmod m$ does not necessarily imply $x \equiv y \pmod m$.

For example, $3.2 \equiv 3.4 \pmod 6$ does not imply $2 \equiv 4 \pmod 6$.

We can cancel the common factor a freely from both sides of the congruence $\pmod m$ provided a is prime to m.

$3. -2 \equiv 2 \pmod 8$, $3.14 \equiv 2 \pmod 8$.

Cancelling the factor 3 which is prime to 8 we get the correct congruence $-2 \equiv 14 \pmod 8$.

Cancellation is allowed however, in some restricted sense which is provided in the following theorem.

Theorem 3.4.4. If $d = gcd(a, m)$ then $ax \equiv ay \pmod{m} \Leftrightarrow x \equiv y \pmod{\frac{m}{d}}$.

Proof. We have $ax - ay = qm$ where q is an integer.

Since $gcd(a, m) = d$, $a = dr$ and $m = ds$ where r and s are integers prime to each other.

Therefore $drx - dry = qds$ or, $x - y = \frac{qs}{r}$.

Since $x - y$ is an integer, $r \mid qs$. r is prime to s and $r \mid qs$ implies $r \mid q$, i.e., $\frac{q}{r}$ is an integer k.

Therefore $x - y = ks$ and this says $x \equiv y \pmod{\frac{m}{d}}$.

Conversely, $x \equiv y \pmod{\frac{m}{d}} \Rightarrow \frac{m}{d} \mid (x - y) \Rightarrow m \mid d(x - y) \Rightarrow m \mid a(x - y) \Rightarrow ax \equiv ay \pmod{m}$.

Corollary. If $ax \equiv ay \pmod{m}$ and $a \mid m$ then $x \equiv y \pmod{\frac{m}{a}}$.

For example, $4.7 \equiv 4.10 \pmod{6}$. Cancellation of 4 from both sides does not give a correct congruence because 4 is not prime to 6. Since $gcd(4, 6) = 2$, we get the correct congruence $7 \equiv 10 \pmod{\frac{6}{2}}$.

Again, $4.7 \equiv 4.10 \pmod{12}$. Since $4 \mid 12$, we get the correct congruence $7 \equiv 10 \pmod{3}$ from the corollary.

Theorem 3.4.5. $x \equiv y \pmod{m_i}$, for $i = 1, 2, \ldots, r \Leftrightarrow x \equiv y \pmod{m}$, where $m = [m_1, m_2, \ldots, m_r]$, the l.c.m. of $m_1, m_2, \ldots, m_r$.

Proof. $x \equiv y \pmod{m_i} \Rightarrow m_i \mid (x - y)$, for $i = 1, 2, \ldots, r$
$\Rightarrow x - y$ is a common multiple of $m_1, m_2, \ldots, m_r$
$\Rightarrow [m_1, m_2, \ldots, m_r] \mid (x - y)$
$\Rightarrow x \equiv y \pmod{m}$.

Conversely, $x \equiv y \pmod{m} \Rightarrow m \mid (x - y)$
$\Rightarrow m_1 m_2 \ldots m_r \mid (x - y)$
$\Rightarrow m_i \mid (x - y)$, for $i = 1, 2, \ldots, r$
$\Rightarrow x \equiv y \pmod{m_i}$, for $i = 1, 2, \ldots, r$.

Corollary. If $x \equiv y \pmod{m_1}$, $x \equiv y \pmod{m_2}$ and m_1, m_2 are relatively prime then $x \equiv y \pmod{m_1 m_2}$.

Theorem 3.4.6. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integral coefficients a_i.

If $a \equiv b \pmod{m}$ then $f(a) \equiv f(b) \pmod{m}$.

Proof. Since $a \equiv b \pmod{m}$, $a^k \equiv b^k \pmod{m}$ where k is a positive integer. Therefore $a_i a^k \equiv a_i b^k \pmod{m}$, where a_i is an integer.

Adding these congruences for $i = 0, 1, 2, \ldots, n$, we have

$a_0 + a_1 a + a_2 a^2 + \cdots + a_n a^n \equiv a_0 + a_1 b + a_2 b^2 + \cdots + a_n b^n \pmod{m}$
or, $f(a) \equiv f(b) \pmod{m}$.

3.4.7. Divisibility tests.

1. Let $n = a_m 10^m + a_{m-1} 10^{m-1} + \cdots + a_2 10^2 + a_1 10 + a_0$ where a_k are integers and $0 \le a_k \le 9, k = 0, 1, \ldots, m$ be the decimal representation of a positive integer n.

Let $S = a_0 + a_1 + \cdots + a_m, T = a_0 - a_1 + \cdots + (-1)^m a_m$. Then

(i) n is divisible by 2 if and only if a_0 is divisible by 2;

(ii) n is divisible by 9 if and only if S is divisible by 9;

(iii) n is divisible by 11 if and only if T is divisible by 11.

Proof. Let us consider the polynomial
$f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$.

(i) We have $10 \equiv 0 \pmod{2}$.
Therefore $f(10) \equiv f(0) \pmod{2}$.
But $f(10) = n$ and $f(0) = a_0$.
Therefore $n - a_0$ is divisible by 2.
Hence n is divisible by 2 if and only if a_0 is divisible by 2.

(ii) We have $10 \equiv 1 \pmod{9}$.
Therefore $f(10) \equiv f(1) \pmod{9}$.
But $f(10) = n$ and $f(1) = S$.
Therefore $n \equiv S \pmod{9}$.
This proves that $N - S$ is divisible by 9.
Hence n is divisible by 9 if and only if S is divisible by 9.

(iii) We have $10 \equiv -1 \pmod{11}$.
Therefore $f(10) \equiv f(-1) \pmod{11}$.
But $f(10) = n$ and $f(-1) = T$.
Therefore $n \equiv T \pmod{11}$.
This proves that $n - T$ is divisible by 11.
Hence n is divisible by 11 if and only if T is divisible by 11.

For example, 35078571 is divisible by 9 since the sum of the digits $3 + 5 + 0 + 7 + 8 + 5 + 7 + 1 (= 36)$ is divisible by 9.

It is also divisible by 11 because the sum $1 - 7 + 5 - 8 + 7 - 0 + 5 - 3 (= 0)$ is divisible by 11.

The number 23572 is divisible by 2, since the integer a_0 in the units place is 2 which is divisible by 2. It is not divisible by 9, since the sum $2 + 3 + 5 + 7 + 2 (= 19)$ is not divisible by 9. It is not divisible by 11, since the sum $2 - 7 + 5 - 3 + 2 (= -1)$ is not divisible by 11.

2. Let $n = a_m (1000)^m + a_{m-1} (1000)^{m-1} + \cdots + a_1 (1000) + a_0$ where a_k are integers and $0 \le a_k \le 999, k = 0, 1, \ldots, m$ be the representation of a

positive integer n.

Let $T = a_0 - a_1 + a_2 - \cdots + (-1)^m a_m$. Then

(i) n is divisible by 7 if and only if T is divisible by 7,

(ii) n is divisible by 13 if and only if T is divisible by 13,

(iii) n is divisible by 11 if and only if T is divisible by 11.

Proof. Let us consider the polynomial

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0.$$

(i) We have $1000 \equiv -1 \pmod 7$ since $1001 = 7.11.13$.

Therefore $f(1000) \equiv f(-1) \pmod 7$.

But $f(1000) = n$ and $f(-1) = T$.

Therefore $n \equiv T \pmod 7$.

This implies $n - T$ is divisible by 7.

Hence n is divisible by 7 if and only if T is divisible by 7.

(ii) and (iii) Similar proofs.

To illustrate, let us consider the number $n = 23146123$. n can be expressed as $23(1000)^2 + 146(1000) + 123$.

n is divisible by 7 because the sum $123 - 146 + 23 = 0$ is divisible by 7.

The same argument proves that n is also divisible by 13 and 11.

Worked Examples.

1. Find the least positive residues in $3^{36} \pmod{77}$.

$3^4 \equiv 4 \pmod{77} \Rightarrow 3^{12} \equiv 4^3 \pmod{77} \equiv -13 \pmod{77}$.

This gives $3^{24} \equiv 169 \pmod{77} \equiv 15 \pmod{77}$.

Therefore $3^{36} \equiv 15.-13 \pmod{77} \equiv 36 \pmod{77}$.

Hence the least positive residue is 36.

2. Use the theory of congruences to prove that $7 \mid 2^{5n+3} + 5^{2n+3}$ for all $n \geq 1$.

$2^{5n+3} + 5^{2n+3} = 8.32^n + 125.25^n$.

$32^n - 25^n \equiv 0 \pmod 7$ for all $n \geq 1$.

Therefore $8.32^n - 8.25^n \equiv 0 \pmod 7$ for all $n \geq 1$.

Also we have $133(25)^n \equiv 0 \pmod 7$ for all $n \geq 1$.

Therefore $8.32^n + 125.25^n \equiv 0 \pmod 7$ for all $n \geq 1$.

This implies $7 \mid 2^{5n+3} + 5^{2n+3}$ for all $n \geq 1$.

3. Prove that $19^{20} \equiv 1 \pmod{181}$.

We have $19^2 \equiv -1 \pmod{181}$, whence

$19^{20} \equiv (-1)^{10} \pmod{181}$, by theorem 3.4.2

or, $19^{20} \equiv 1 \pmod{181}$.

4. Prove that $3.4^{n+1} \equiv 3 \pmod 9$ for all positive integers n.

$3.4^{n+1} = 12.4^n = 9.4^n + 3.4^n$

$3.4^n = 12.4^{n-1} = 9.4^{n-1} + 3.4^{n-1}$

$\cdots \quad \cdots \quad \cdots$

$3.4^2 = 12.4 = 9.4 + 3.4$

$3.4 = 12 = 9 + 3.$

Therefore $3.4^{n+1} = 9(1 + 4 + 4^2 + \cdots + 4^n) + 3$.

Hence $3.4^{n+1} \equiv 3 \pmod 9$.

5. Find the remainder when $1! + 2! + 3! + \cdots + 50!$ is divided by 15.

$5! \equiv 0 \pmod{15}$ and $(5+n)! \equiv 0 \pmod{15}$ for any positive integer n.

Therefore $1! + 2! + 3! + \cdots + 50! \equiv (1! + 2! + 3! + 4!) \pmod{15}$.

$\equiv 33 \pmod{15}$

$\equiv 3 \pmod{15}$.

3.4.8. Linear congruence.

Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n (n \geq 1)$ be a polynomial with integer coefficients $a_0, a_1, \ldots, a_n$ with $a_0 \not\equiv 0 \pmod m$. Then $f(x) \equiv 0 \pmod m$ is said to be a *polynomial congruence* (mod m) of *degree* n.

If there exists an integer x_0 such that $f(x_0) \equiv 0 \pmod m$, then x_0 is said to be *a solution* of the congruence.

By earlier theorems, if x_1 be any integer satisfying $x_1 \equiv x_0 \pmod m$, then we also have $f(x_1) \equiv 0 \pmod m$, showing that x_1 is another solution of the congruence.

Thus if one solution be found then infinitely many solutions can be obtained, but all these solutions belong to the same x_0-residue class modulo m and they are not counted as different solutions.

Two solutions x_1, x_2 of $f(x) \equiv 0 \pmod m$ are said to be *distinct* solutions if $x_1 \not\equiv x_2 \pmod m$.

Therefore, by the number of solutions of a congruence (mod m) we mean the number of solutions *incongruent in pairs*.

For example, let us consider the congruence $x^2 \equiv 1 \pmod 8$. $x = 1$

is a solution of the congruence and all solutions congruent to 1 (mod 8), i.e., $x = 1 + 8k$, k being an integer, are solutions of the congruence.

$x = 3$ is a solution of the congruence and all solutions congruent to 3 (mod 8), i.e., $x = 3 + 8k$, k being an integer are solutions of the congruence.

Similarly, $x = 5$, $x = 7$ are solutions of the congruence.

These four solutions of the congruence are distinct, because no two of the solutions are congruent modulo 8.

There cannot be more than m distinct solutions of the congruence, since there are only m different residue classes. If m is small it is an easy job to find all the distinct solutions by direct substitution $x = 1$, $x = 2$, ... , $x = m - 1$.

There are many points of difference between a polynomial congruence modulo a positive integer $m > 1$ and the polynomial equation over the field of complex numbers.

A congruence may have no solution. For example, the congruence $x^2 \equiv 3 \pmod 5$ has no solution which can be established by directly verifying that none of $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = 4$ satisfies the congruence. In contrast, a polynomial equation has always a solution.

A congruence may have more distinct solutions than its degree. For example, the congruence $x^2 \equiv 1 \pmod 8$ has four distinct solutions $x = 1, x = 3$, $x = 5$, $x = 7$. In contrast, a polynomial equation of degree m over the complex field has exactly m solutions.

There is an explicit method of solving a congruence of any degree modulo a positive integer $m > 1$. [just by substitution of each of the integers $1, 2, \ldots, m - 1$, in turn.] But there is no such explicit method for solving a polynomial equation of degree greater than 4.

Definition.

A polynomial congruence of degree 1 is said to be a *linear congruence*. The general form of a linear congruence modulo a positive integer $m > 1$ is $ax \equiv b \pmod m$, where $a \not\equiv 0 \pmod m$.

An integer c is said to be a *solution* of the linear congruence $ax \equiv b \pmod m$ if $ac \equiv b \pmod m$.

Theorem 3.4.9. If x_1 be a solution of the linear congruence $ax \equiv b \pmod m$ and if $x_2 \equiv x_1 \pmod m$, then x_2 is also a solution of the congruence.

Proof. x_1 is a solution $\Rightarrow ax_1 \equiv b \pmod m$.

$x_2 \equiv x_1 \pmod m \Rightarrow ax_2 \equiv ax_1 \pmod m$

$\Rightarrow ax_2 \equiv b \pmod m$.

This shows that x_2 is a solution of the congruence $ax \equiv b \pmod m$.

Note. If x_1 be a solution of the congruence $ax \equiv b \pmod m$ then $x_1 + \lambda m$ is also a solution for $\lambda = 0, \pm 1, \pm 2, \ldots$. All these solutions belong to one residue class modulo m and these are not counted as different solutions.

Theorem 3.4.10. If $gcd(a, m) = 1$, then the linear congruence $ax \equiv b \pmod m$ has a *unique* solution.

Proof. Since $gcd(a, m) = 1$, there exist integers u, v such that $au + mv = 1$. Therefore $a(bu) + m(bv) = b$. This gives $a(bu) \equiv b \pmod m$.

This shows that $x = bu$ is a solution of the congruence $ax \equiv b \pmod m$.

Let x_1, x_2 be solutions of the congruence $ax \equiv b \pmod m$.

Then $ax_1 \equiv b \pmod m$ and $ax_2 \equiv b \pmod m$.

This implies $ax_1 \equiv ax_2 \pmod m \Rightarrow x_1 \equiv x_2 \pmod m$, since $gcd(a, m) = 1$.

This proves that the congruence has a unique solution.

Note. The solutions are $x = bu + \lambda m$, where $\lambda = 0, \pm 1, \pm 2, \ldots$ and they all belong to *one and only one* residue class modulo m.

Theorem 3.4.11. If $gcd(a, m) = d$, then the linear congruence $ax \equiv b \pmod m$ has no solution if d is not a divisor of b.

If d be divisor of b, then the linear congruence $ax \equiv b \pmod m$ has d incongruent solutions $\pmod m$.

Proof. Let $ax \equiv b \pmod m$ has a solution $x = u$. Then $au \equiv b \pmod m$ and this implies $m \mid (b - au)$.

$d \mid m \Rightarrow d \mid (b - au)$. $d \mid a$ and $d \mid (b - au) \Rightarrow d$ is a divisor of b.

Contrapositively, d is not a divisor of b implies $ax \equiv b \pmod m$ has no solution.

Second part. $d \mid b$. For an integer u, $au \equiv b \pmod m$ holds if and only if $\frac{a}{d}u \equiv \frac{b}{d} \pmod{\frac{m}{d}}$, by Theorem 3.4.4.

$gcd(\frac{a}{d}, \frac{m}{d}) = 1$ and therefore the congruence $\frac{a}{d}u \equiv \frac{b}{d} \pmod{\frac{m}{d}}$ has just one solution $u = x_1 \pmod{\frac{m}{d}}$.

In other words, the solution of the congruence $\frac{a}{d}u \equiv \frac{b}{d} \pmod{\frac{m}{d}}$ are the integers $u \equiv x_1 \pmod{\frac{m}{d}}$, i.e., $u = x_1 + \frac{m}{d}t$, $t = 0, \pm 1, \pm 2, \ldots$

If t assumes the values $0, 1, 2, \ldots, d-1$, then u assumes d values $x_1, x_1 + \frac{m}{d}, x_1 + \frac{2m}{d}, \cdots, x_1 + \frac{(d-1)m}{d}$ ... (i)

We now show that the integers in the list (i) are incongruent modulo m, while each of all other solutions (corresponding to the values of t other than $0, 1, \ldots, d-1$) is congruent to some one of the integers in (i).

$x_1 + t_1\frac{m}{d} \equiv x_1 + t_2\frac{m}{d} \pmod{m}$, where $0 \le t_1 < t_2 \le d-1$ gives $t_1\frac{m}{d} \equiv t_2\frac{m}{d} \pmod{m}$.

$\gcd(\frac{m}{d}, m) = \frac{m}{d} \Rightarrow t_1 \equiv t_2 \pmod{d} \Rightarrow d \mid t_2 - t_1$.

This is an impossibility, because $0 < t_2 - t_1 < d$.

Thus all solutions in the list (i) are incongruent modulo m.

Let any other solution be $x_1 + t_j\frac{m}{d}$, where t_j is an integer other than $0, 1, \ldots, d-1$.

By Divison algorithm we can write $t_j = qd + r$, where q and r are integers and $0 \le r \le d-1$.

Then $x_1 + t_j\frac{m}{d} = x_1 + (qd + r)\frac{m}{d} \equiv x_1 + r\frac{m}{d} \pmod{m}$.

Since $0 \le r \le d-1$, $x_1 + t_j\frac{m}{d}$ is one of the solutions listed in (i).

Thus the congruence $ax \equiv b \pmod{m}$ has d incongruent solutions listed in (i).

This completes the proof.

Note. The solutions belong to a single residue class modulo $\frac{m}{d}$ and this is the union of d distinct residue classes modulo m. The residue class $\bar{i}$ modulo $\frac{m}{d}$ is the union of d distinct residue classes $\bar{i}, \overline{i + \frac{m}{d}}, \overline{i + \frac{2m}{d}}, \ldots, \overline{i + \frac{(d-1)m}{d}}$ modulo m.

For example, the residue class $\bar{1}$ modulo 5 is the union of the three distinct residue classes $\bar{1}, \bar{6}, \bar{11}$ modulo 15.

Worked Examples.

1. Solve the linear congruence $5x \equiv 3 \pmod{11}$.

$\gcd(5, 11) = 1$. Hence the congruence has a unique solution.

Since $\gcd(5, 11) = 1$, there exist integers u, v such that $5u + 11v = 1$.

Here $u = -2, v = 1$. Therefore $5.(-2) + 11.1 = 1$ and this implies $5.(-2) \equiv 1 \pmod{11}$. Therefore $5.(-6) \equiv 3 \pmod{11}$.

Hence $x = -6$ is a solution.

All solutions are $x \equiv -6 \pmod{11}$, i.e., $x \equiv 5 \pmod{11}$.

All the solutions are congruent to 5 (mod 11) and therefore the given congruence has a unique solution.

2. Solve the linear congruence $15x \equiv 9 \pmod{18}$.

$\gcd(15, 18) = 3$ and $3 \mid 9$. Therefore the given congruence has a solution. The given congruence is equivalent to $5x \equiv 3 \pmod{6}$.

$\gcd(5, 6) = 1$. Hence the congruence $5x \equiv 3 \pmod{6}$ has a unique solution.

Since $\gcd(5, 6) = 1$, there exist integers u, v such that $5u + 6v = 1$.

Here $u = -1, v = 1$. Therefore $5.(-1) + 6.1 = 1$ and this implies $5.(-1) \equiv 1 \pmod{6}$. Therefore $5.(-3) \equiv 3 \pmod{6}$. Hence $x = -3$ is a solution of the congruence $5x \equiv 3 \pmod{6}$.

There are three incongruent solutions of the given congruence. They are $x = -3, -3 + 6, -3 + 12$ modulo 18, i.e., $x \equiv -3, 3, 9 \pmod{18}$.

System of linear congruences.

Let us consider the linear congruences

$a_1x \equiv b_1 \pmod{m_1}$, $a_2x \equiv b_2 \pmod{m_2}$, $\cdots$, $a_rx \equiv b_r \pmod{m_r}$

and let us enquire if it be possible to have a simultaneous solution of the congruences. In that case each individual congruence must have a solution.

Let $\gcd(a_i, m_i) = d_i$ for $i = 1, 2, \ldots, r$. Then d_i must be a divisor of b_i for each i.

Cancelling d_i from the ith equation, the system reduces to

$a_1'x \equiv b_1' \pmod{m_1'}$, $a_2'x \equiv b_2' \pmod{m_2'}$, $\cdots$, $a_r'x \equiv b_r' \pmod{m_r'}$,

where $d_ia_i' = a_i$, $d_ib_i' = b_i$, $\ldots$, $d_im_i' = m_i$ and $\gcd(a_i', m_i') = 1$ for $i = 1, 2, \ldots, r$.

Each individual congruence has a unique solution of the form $x_i \equiv c_i \pmod{m_i}$.

Thus the problem is reduced to one of finding a common solution of the system

$x \equiv c_1 \pmod{m_1}$, $x \equiv c_2 \pmod{m_2}$, $\cdots$, $x \equiv c_r \pmod{m_r}$.

The kind of problems that can be reduced to a system of linear congruences was found in Chinese literature as early as first century A.D..In later periods such problems were also found in other countries. Because of their antiquity, this type of problem goes by the name of "Chinese remainder theorem".

The method of congruence that is used to state the problem and to make the proof in a concise form was unknown to the ancients.

Theorem 3.4.12. Chinese remainder theorem.

Let $m_1, m_2, \ldots, m_r$ be positive integers such that $gcd(m_i, m_j) = 1$ for $i \neq j$ and $c_1, c_2, \ldots, c_r$ be any integers. Then the system of linear congruences

$x \equiv c_1 \pmod{m_1}$, $x \equiv c_2 \pmod{m_2}$, $\cdots$, $x \equiv c_r \pmod{m_r}$,

has a simultaneous solution which is unique modulo $m_1 m_2 \ldots m_r$. [i.e., if x_0 be a solution then $x = x_0 + k(m_1 m_2 \ldots m_r)$ is also a solution, where k is an integer.]

Proof. Let $m = m_1 m_2 \ldots m_r$. Let $M_k = \frac{m}{m_k}, k = 1, 2, \ldots, r$.

Then $gcd(M_k, m_k) = 1$ for $k = 1, 2, \ldots, r$.

This implies that the linear congruence $M_k x \equiv 1 \pmod{m_k}$ has a unique solution modulo m_k. Let x_k be the solution.

Then $M_k x_k \equiv 1 \pmod{m_k}$ and clearly, $M_k x_k \equiv 0 \pmod{m_j}$ for $j \neq k$.

Therefore $c_k M_k x_k \equiv c_k \pmod{m_k}$ and $c_k M_k x_k \equiv 0 \pmod{m_j}$ for $j \neq k$.

Let us consider the integer $x_0 = c_1 M_1 x_1 + c_2 M_2 x_2 + \cdots + c_r M_r x_r$.

$x_0 \equiv c_1 \pmod{m_1}$, since $c_1 M_1 x_1 \equiv c_1 \pmod{m_1}$ and $c_i M_i x_i \equiv 0 \pmod{m_1}$ for $i \neq 1$.

Similarly, $x_0 \equiv c_2 \pmod{m_2}$ ... $x_0 \equiv c_r \pmod{m_r}$.

This shows that x_0 is a solution of the given system of congruences.

Let x' be any solution of the system of congruences.

Then $x' \equiv c_k \pmod{m_k}$ for $k = 1, 2, \ldots, r$.

$x' \equiv c_k \pmod{m_k}$ and $x_0 \equiv c_k \pmod{m_k} \Rightarrow x' \equiv x_0 \pmod{m_k}$ for $k = 1, 2, \ldots, r$.

Consequently, $x' \equiv x_0 \pmod{m_1 m_2 \ldots m_r}$ [Theorem 3.4.5]

This shows that the solution of the system is unique modulo $m_1 m_2 \ldots m_r$.

Note. In the Chinese remainder theorem, the hypothesis that $m_1, m_2, \ldots, m_r$ should be pairwise relatively prime is essential. If this condition be not satisfied there may not exist a solution of the system of congruences.

For example, the simultaneous congruences $x \equiv 1 \pmod 4$ and $x \equiv 3 \pmod 8$ has no solution, while the system $x \equiv 3 \pmod{10}$ and $x \equiv 8 \pmod{15}$ has the unique solution $x \equiv 23 \pmod{30}$.

Worked Examples (continued).

3. Solve the system of linear congruences

$x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$.

3, 5 and 7 are pairwise prime to each other. Let $m = 3.5.7 = 105$.

Let $M_1 = \frac{m}{3} = 35$, $M_2 = \frac{m}{5} = 21$, $M_3 = \frac{m}{7} = 15$.

Then $gcd(M_1, 3) = 1, gcd(M_2, 5) = 1, gcd(M_3, 7) = 1$.

$gcd(35, 3) = 1$. Therefore the linear congruence $35x \equiv 1 \pmod 3$ has a unique solution and the solution is $x \equiv 2 \pmod 3$.

$gcd(21, 5) = 1$. Therefore the linear congruence $21x \equiv 1 \pmod 5$ has a unique solution and the solution is $x \equiv 1 \pmod 5$.

$gcd(15, 7) = 1$. Therefore the linear congruence $15x \equiv 1 \pmod 7$ has a unique solution and the solution is $x \equiv 1 \pmod 7$.

$x_0 = 1.(35.2) + 2.(21.1) + 3.(15.1) = 157$ is a solution.

The solution of the given system is $x \equiv 157 \pmod{105}$, which is equivalent to $x \equiv 52 \pmod{105}$.

4. Find four consecutive integers divisible by 3,4,5,7 respectively.

Let $n, n+1, n+2, n+3$ be four consecutive integers divisible by 3,4,5,7 respectively. Then $n \equiv 0 \pmod 3$, $n+1 \equiv 0 \pmod 4$, $n+2 \equiv 0 \pmod 5$, $n+3 \equiv 0 \pmod 7$.

We are to solve simultaneous linear congruences $n \equiv 0 \pmod 3$, $n \equiv 3 \pmod 4$, $n \equiv 3 \pmod 5$, $n \equiv 4 \pmod 7$.

3, 4, 5 and 7 are pairwise prime to each other. Let $m = 3.4.5.7$.

Let $M_1 = \frac{m}{3} = 140$, $M_2 = \frac{m}{4} = 105$, $M_3 = \frac{m}{5} = 84$, $M_4 = \frac{m}{7} = 60$.

Since $gcd(3, M_1) = gcd(3, 140) = 1$, the linear congruence $140x \equiv 1 \pmod 3$ has a unique solution (mod 3) and the solution is $x = 2$.

Since $gcd(4, M_2) = gcd(4, 105) = 1$, the linear congruence $105x \equiv 1 \pmod 4$ has a unique solution (mod 4) and the solution is $x = 1$.

Since $gcd(5, M_3) = gcd(5, 84) = 1$, the linear congruence $84x \equiv 1 \pmod 5$ has a unique solution (mod 5) and the solution is $x = 4$.

Since $gcd(7, M_4) = gcd(7, 60) = 1$, the linear congruence $60x \equiv 1 \pmod 7$ has a unique solution (mod 7) and the solution is $x = 2$.

Therefore $x_0 = 0.140.2 + 3.105.1 + 3.84.4 + 4.60.2 = 315 + 1008 + 480 = 1803$ is a solution and the solution is unique modulo 420.

$x_0 = 123 \pmod{420}$. Therefore the consecutive integers are $n, n+1, n+2, n+3$, where $n = 123 + 420t$, $t = 0, \pm 1, \pm 2, \ldots$

5. Solve the linear congruence $32x \equiv 79 \pmod{1225}$ by applying Chinese remainder theorem.

$1225 = 5^2.7^2$ and $gcd(5^2, 7^2) = 1$. The given problem is equivalent to finding a simultaneous solution of the congruences $32x \equiv 79 \pmod{25}$ and $32x \equiv 79 \pmod{49}$, since $a \equiv b \pmod{m_1}$ and $a \equiv b \pmod{m_2} \Leftrightarrow a \equiv b \pmod{[m_1, m_2]}$.

$32x \equiv 79 \pmod{25}$ is equivalent to $7x \equiv 4 \pmod{25}$.

$gcd(7, 25) = 1$. We have $7.(-7) + 25.2 = 1$. Therefore $7.(-7) \equiv 1 \pmod{25}$ and therefore $7.(-28) \equiv 4 \pmod{25}$. This shows that $7x \equiv 4 \pmod{25}$ is equivalent to $x \equiv -28 \pmod{25}$, i.e., $x \equiv 22 \pmod{25}$.

$32x \equiv 79 \pmod{49}$ is equivalent to $32x \equiv 30 \pmod{49}$ which is again equivalent to $16x \equiv 15 \pmod{49}$.

$gcd(16, 49) = 1$. We have $16.(-3) + 49.1 = 1$. Therefore $16.(-3) \equiv 1 \pmod{49}$ and therefore $16.(-45) \equiv 15 \pmod{49}$. This shows that $16x \equiv 15 \pmod{49}$ is equivalent to $x \equiv -45 \pmod{49}$, i.e., $x \equiv 4 \pmod{49}$.

We now solve the system of congruences

$x \equiv 22 \pmod{25}$, $x \equiv 4 \pmod{49}$.

Let $m = 25.49$. Let $M_1 = \frac{m}{25} = 49$, $M_2 = \frac{m}{49} = 25$.

Since $gcd(M_1, 25) = 1$, the congruence $49x \equiv 1 \pmod{25}$ has a unique solution.

We have $49.(-1) + 25.2 = 1$. Therefore $49.(-1) \equiv 1 \pmod{25}$ and therefore the unique solution is $x_1 \equiv 24 \pmod{25}$.

Since $gcd(M_2, 49) = 1$, the congruence $25x \equiv 1 \pmod{49}$ has a unique solution. Proceeding similarly, the unique solution is $x_2 \equiv 2 \pmod{49}$.

A solution of the system is $x_0 = 22.(49.24) + 4.(25.2) = 26072$.

The solution of the given congruence is $x \equiv 26072 \pmod{1225}$, which is equivalent to $x \equiv 347 \pmod{1225}$.

Theorem 3.4.13. Let m_1, m_2 be positive integers and a_1, a_2 be any integers. Then the system of linear congruences

$x \equiv a_1 \pmod{m_1}$, $x \equiv a_2 \pmod{m_2}$

will have a simultaneous solution if and only if $a_1 - a_2$ is divisible by $gcd(m_1, m_2)$ and if this condition be satisfied the solution is unique modulo $lcm(m_1, m_2)$. [i.e., the solutions form a single congruence class modulo $[m_1, m_2]$.]

Proof. Let $m_{12} = gcd(m_1, m_2)$ and $m_1 = m_{12}d_1, m_2 = m_{12}d_2$, where $gcd(d_1, d_2) = 1$.

Let x_0 be a simultaneous solution. Then $x_0 \equiv a_1 \pmod{m_1}$, $x_0 \equiv a_2 \pmod{m_2}$. Therefore $m_1|(x_0 - a_1)$ and $m_2|(x_0 - a_2)$ and this implies $m_{12}|(x_0 - a_1)$ and $m_{12}|(x_0 - a_2)$.

Therefore $m_{12}|(x_0 - a_2) - (x_0 - a_1)$, i.e, $gcd(m_1, m_2)|(a_1 - a_2)$.

Conversely, let $m_{12}|(a_1 - a_2)$. Then $a_1 = a_2 + cm_{12}$ for some integer c.

The general solution of the first congruence has the form

$$\begin{aligned} x &= a_1 + m_1 s, \text{ where } s \text{ is an integer} \\ &= (a_2 + cm_{12}) + d_1 m_{12} s, \\ &= a_2 + m_{12}(c + d_1 s), \text{ where } s \text{ is an integer} \quad \ldots \text{ (i)} \end{aligned}$$

The general solution of the second congruence has the form

$$\begin{aligned} x &= a_2 + m_2 t, \text{ where } t \text{ is an integer} \\ &= a_2 + m_{12} d_2 t, \text{ where } t \text{ is an integer} \quad \ldots \text{ (ii)} \end{aligned}$$

Since $gcd(d_1, d_2) = 1$, $pd_1 + qd_2 = 1$ for some integers p and q. Therefore $pcd_1 + qcd_2 = c$ and $c + d_1 s = d_1(pc + s) + qcd_2$.

If we choose s such that $pc + s = d_2 u$, where u is an integer then $c + d_1 s = d_2(ud_1 + qc) = d_2 v$, where v is an integer and in that case the solutions (i) and (ii) become identical which implies that the given congruences have a simultaneous solution.

The solution takes the form

$$\begin{aligned} x &= a_1 + m_1 s = a_1 + m_1(-pc + d_2 u), \text{ where } u \text{ is an integer} \\ &= a_1 - pcm_1 + m_1 d_2 u \\ &= a_1 - pcm_1 + lcm(m_1, m_2)u, \text{ since } m_1 d_2 = \tfrac{m_1 m_2}{gcd(m_1, m_2)} = [m_1, m_2]. \end{aligned}$$

Since u is an integer, the solution is unique modulo $lcm(m_1, m_2)$.

6. Show that the congruences $x \equiv 11 \pmod{15}$, $x \equiv 6 \pmod{35}$ have a simultaneous solution. Solve the system.

Here the moduli of the congruences 15 and 35 are not prime to each other. $gcd(15, 35) = 5$ and $11 - 6$ is divisible by $gcd(15, 35)$. Therefore the congruences have a simultaneous solution.

The general solution of the first congruence has the form

$$\begin{aligned} x &= 11 + 15s, \text{ where } s \text{ is an integer} \\ &= 6 + 5(1 + 3s) \quad \ldots \text{ (i)} \end{aligned}$$

The general solution of the second congruence has the form

$x = 6 + 35t = 6 + 5.7t$, where t is an integer ... (ii)

If $s = 2 + 7u$, where u is an integer, then $1 + 3s$ is a multiple of 7 and in that case (i) and (ii) become identical.

The common solution is

$x = 6 + 5[1 + 3(2 + 7u)] = 41 + 105u$, where u is an integer.

Therefore the solution is given by $x \equiv 41 \pmod{105}$.

Another method

The congruence $x \equiv 11 \pmod{15}$ is equivalent to two simultaneous congruences $x \equiv 11 \pmod 3$, $x \equiv 11 \pmod 5$.

The congruence $x \equiv 6 \pmod{35}$ is equivalent to two simultaneous congruences $x \equiv 6 \pmod 7$, $x \equiv 6 \pmod 5$.

Thus the given system is equivalent to four simultaneous congruences $x \equiv 11 \pmod 3$, $x \equiv 11 \pmod 5$, $x \equiv 6 \pmod 7$, $x \equiv 6 \pmod 5$.

It is observed that the congruences $x \equiv 11 \pmod 5$ and $x \equiv 6 \pmod 5$ are consistent, because each is equivalent to $x \equiv 1 \pmod 5$.

The congruence $x \equiv 11 \pmod 3$ is equivalent to $x \equiv 2 \pmod 3$.

Thus the given system is equivalent to three simultaneous congruences $x \equiv 2 \pmod 3$, $x \equiv 1 \pmod 5$, $x \equiv 6 \pmod 7$ of which the moduli are pairwise prime to each other.

Using the method discussed in the theorem,

$x_0 = 2.(35.2) + 1.(21.1) + 6.(15.1) = 251$ is a solution.

The solution of the given system is $x \equiv 251 \pmod{105}$, which is equivalent to $x \equiv 41 \pmod{105}$.

There is an extension of the above theorem to a finite set of linear congruences. The theorem is stated below without proof.

Theorem 3.4.14. Let $m_1, m_2, \ldots, m_r$ be positive integers and $a_1, a_2, \ldots, a_r$ be any integers. Then the system of linear congruences

$$x \equiv a_1 \pmod{m_1},\ x \equiv a_2 \pmod{m_2}, \ldots, x \equiv a_r \pmod{m_r}$$

will have a simultaneous solution if and only if $a_i - a_j$ is divisible by $gcd(m_i, m_j)$ whenever $i \neq j$; and if this condition be satisfied the solution is unique modulo $lcm(m_1, m_2, \ldots, m_r)$. [i.e., the solutions form a single congruence class modulo $lcm(m_1, m_2, \ldots, m_r)$.]

For example, the system of congruences $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 4 \pmod 6$ has no solution, since $4 - 2$ is not divisible by $gcd(3, 6)$.

The solution of the congruences $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$ is $x \equiv 8 \pmod{15}$. Therefore the given system reduces to $x \equiv 8 \pmod{15}$, $x \equiv 4 \pmod 6$ and this system has no solution, since $8 - 4$ is not divisible by $gcd(15, 6)$.

Exercises 3B

1. List all integers x in $1 \leq x \leq 100$ that satisfy
   (i) $x \equiv 3 \pmod{17}$. (ii) $x \equiv 10 \pmod{17}$.

2. Prove that
   (i) an integer n is divisible by 3 if and only if the sum of its digits is divisible by 3.
   (ii) an integer n is divisible by 5^k if and only if the number formed by the last k digits is divisible by 5^k.
   (iii) an integer n is divisible by 2^k if and only if the number formed by the last k digits is divisible by 2^k.

[Hint. (ii) If the last k digits form the number a, then $n = b.10^k + a$. $n \equiv a \pmod{5^k}$.]

3. Find the missing digit in the number $23104 * 791$, if
   (i) it is divisible by 7; (ii) it is divisible by 9;
   (iii) it is divisible by 11; (iv) it is divisible by 13.

4. A positive integer n is expressed in the form $10a + b$.
   (i) Prove that n is divisible by 7 if $a - 2b$ is divisible by 7.
   (ii) Prove that n is divisible by 17 if $a - 5b$ is divisible by 17.
   (iii) Prove that n is divisible by 19 if $a + 2b$ is divisible by 19.

[Hint. (iii) $(10a + b) + 9(a + 2b) \equiv 0 \pmod{19}$.]

5. Use the theory of congruences to prove that
   (i) $17 \mid (2^{3n+1} + 3.5^{2n+1})$ for all integers $n \geq 1$,
   (ii) $23 \mid (2^{5n+1} + 7.3^{2n+1})$ for all integers $n \geq 1$,
   (iii) $43 \mid (6^{n+2} + 7^{2n+1})$ for all integers $n \geq 1$.

6. Use the theory of congruences to establish that
   (i) $73 \mid (2^{36} - 1)$, (ii) $89 \mid (2^{44} - 1)$.

7. Use the theory of congruences to find the remainder when the sum
   (i) $1^3 + 2^3 + 3^3 + \cdots + 99^3$ is divided by 3;
   (ii) $1^5 + 2^5 + 3^5 + \cdots + 100^5$ is divided by 5.

8. (i) If $ac \equiv bc \pmod m$ and $gcd(c, m) = 1$, prove that $a \equiv b \pmod m$.
   (ii) If $a \equiv b \pmod m$ and $a \equiv b \pmod n$ where $gcd(m, n) = 1$, prove that $a \equiv b \pmod{mn}$.
   (iii) If $a \equiv b \pmod m$ and $a \equiv c \pmod n$, prove that $b \equiv c \pmod d$, where $d = gcd(m, n)$.

9. Solve the linear congruences.

(i) $4x \equiv 3 \pmod 5$, (ii) $7x \equiv 3 \pmod{15}$,

(iii) $20x \equiv 10 \pmod{35}$, (iv) $28x \equiv 63 \pmod{105}$.

10. Solve the system of linear congruences.

(i) $x \equiv 3 \pmod 5$, $x \equiv 4 \pmod 7$;

(ii) $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 4 \pmod 7$;

(iii) $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$, $x \equiv 5 \pmod 8$.

11. (i) Find the least positive integer which leaves remainders 2, 3 and 2 when divided by 3, 4 and 5 respectively.

(ii) Find the least positive integer which leaves remainders 2, 3 and 4 when divided by 3, 5 and 11 respectively.

12. (i) Find three consecutive positive integers in ascending order, divisible by 5, 7 and 9 respectively.

(ii) Find four consecutive positive integers in ascending order, divisible by 4, 5, 7 and 9 respectively.

13. Solve the linear congruences by Chinese remainder theorem.

(i) $25x \equiv 4 \pmod{63}$, (ii) $89x \equiv 25 \pmod{108}$.

14. Show that the system of linear congruences has a simultaneous solution and solve the system.

(i) $x \equiv 2 \pmod 9$, $x \equiv 5 \pmod 6$;

(ii) $x \equiv 2 \pmod{15}$, $x \equiv 7 \pmod{10}$.

3.5. Phi function.

Definition. The function ϕ, called *Euler's phi function* (Euler's *totient function*), is defined for all positive integers by $\phi(1) = 1$ and for $n > 1$, $\phi(n)$ = the number of positive integers less than n and prime to n.

For example, let $n = 8, 20$.

The positive integers less than 8 and prime to 6 are $1, 3, 5, 7$. Therefore $\phi(8) = 4$.

The positive integers less than 20 and prime to 20 are $1, 3, 7, 9, 11, 13, 17, 19$. Therefore $\phi(20) = 8$.

If p be a prime then every positive integer less than p is prime to p. Therefore $\phi(p) = p - 1$.

Note. The function ϕ is a number-theoretic function.

Theorem 3.5.1. The function ϕ is a multiplicative function. That is, if m and n be relatively prime integers, then $\phi(mn) = \phi(m)\phi(n)$.

First we prove the following lemmas.

Lemma 1. a is prime to mn if and only if a is prime to m and a is prime to n.

Proof. Let a be prime to mn and $d = gcd(a, m)$. Then $d \mid a$ and $d \mid m$ and this implies $d \mid mn$.

Therefore $gcd(a, mn) \geq d$, but $gcd(a, mn) = 1$ by assumption.

Hence $d = 1$ proving that a is prime to m. By similar arguments, a is prime to n.

Conversely, let a be prime to m and a be prime to n. Since a is prime to m, there exist integers u and v such that $au + mv = 1$. Since a is prime to n, there exist integers p and q such that $ap + nq = 1$.

We have $anuq + mnvq = nq = 1 - ap$

or, $a(nuq + p) + mn(vq) = 1$.

Since $nuq + p$ and vq are integers, it follows that a is prime to mn.

Lemma 2. If r be the residue of a modulo n and r is prime to n then a is prime to n.

Since $gcd(qn + r, n) = gcd(r, n)$, the lemma follows.

Lemma 3. If c be an integer and a is prime to n then the number of integers in the set $\{c, c+a, c+2a, \ldots, c+(n-1)a\}$ that are prime to n is $\phi(n)$.

Proof. No two integers of the set are congruent modulo n, because

$$c + sa \equiv c + ta \pmod n \qquad 0 \leq s < t \leq n-1$$

$$\Rightarrow s \equiv t \pmod n, \text{ a contradiction.}$$

Therefore the set of integers is congruent modulo n to $0, 1, 2, \ldots, n-1$ in some order. Since the number of integers among $0, 1, 2, \ldots n-1$ that are prime to n is $\phi(n)$, the lemma follows.

Proof of the theorem.

Since $\phi(1) = 1$, the theorem is trivially true when m or n equals 1. Let us assume $m > 1, n > 1$. We arrange mn integers in n rows and m columns as follows:

1	2	$\ldots$	r	$\ldots$	m
$m+1$	$m+2$	$\ldots$	$m+r$	$\ldots$	$2m$
$2m+1$	$2m+2$	$\ldots$	$2m+r$	$\ldots$	$3m$
$\ldots$	$\ldots$		$\ldots$		
$(n-1)m+1$	$(n-1)m+2$	$\ldots$	$(n-1)m+r$	$\ldots$	nm.

The number of integers among these that are prime to mn is $\phi(mn)$. By lemma 1, these integers are both prime to m and n.

The number of integers in the first row that are prime to m is $\phi(m)$. By lemma 2, each integer in the column of $r(1 \le r \le m)$ is prime to m if r is prime to m. So there are $\phi(m)$ columns of integers prime to m.

Again by lemma 3, each column in the arrangement contains $\phi(n)$ integers prime to n. Therefore the number of integers those are prime to m as well as n is $\phi(m).\phi(n)$.
This completes the proof.

Note. $\phi(m_1 m_2 \dots m_r) = \phi(m_1)\phi(m_2)\dots\phi(m_r)$, where $m_1, m_2, \dots, m_r$ are pairwise prime.

Theorem 3.5.2. If p be a prime and k be a positive integer,

$$\phi(p^k) = p^k(1 - \tfrac{1}{p}).$$

Proof. The positive integers $\le p^k$ which are not prime to p^k are $p, 2p, 3p, \dots, (p^{k-1})p$. Therefore the number of positive integers less than p^k and prime to p^k is $p^k - p^{k-1}$.

Hence $\phi(p^k) = p^k - p^{k-1} = p^k(1 - \frac{1}{p})$.

Theorem 3.5.3. If $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ where $p_1, p_2, \dots, p_r$ are prime to one another, $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})\dots(1 - \frac{1}{p_r})$.

Proof. $\phi(p_1^{a_1}) = p_1^{a_1}(1 - \frac{1}{p_1}).\ \phi(p_2^{a_2}) = p_2^{a_2}(1 - \frac{1}{p_2}), \dots$

Since $p_1^{a_1}$ and $p_2^{a_2}$ are prime to each other,

$\phi(p_1^{a_1} p_2^{a_2}) = \phi(p_1^{a_1})\phi(p_2^{a_2}) = p_1^{a_1} p_2^{a_2}(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})$.

Since $p_3^{a_3}$ is prime to $p_1^{a_1} p_2^{a_2}$, $\phi(p_3^{a_3} p_1^{a_1} p_2^{a_2}) = \phi(p_3^{a_3}).\phi(p_1^{a_1} p_2^{a_2})$

$= p_3^{a_3}(1 - \frac{1}{p_3}).p_1^{a_1} p_2^{a_2}(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})$

$= p_1^{a_1} p_2^{a_2} p_3^{a_3}(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})(1 - \frac{1}{p_3})$.

Proceeding with similar arguments we have
$\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})\dots(1 - \frac{1}{p_r})$.

To illustrate the theorem, let $n = 12$. The integers less than 12 and prime to 12 are $1, 5, 7, 11$. Here $\phi(12) = \phi(2^2.3) = 12(1 - \frac{1}{2})(1 - \frac{1}{3}) = 4$.

Some properties of phi function.

1. If $n > 1$, the sum of all positive integers less than n and prime to n is $\frac{1}{2}n.\phi(n)$.

Proof. Let a be a positive integer less than n and prime to n. Then $au + nv = 1$ for some integers u and v.

Therefore $n(v+u) + (n-a)(-u) = 1$. Since $u+v$ and $-u$ are integers, it follows that $n - a$ is also a positive integer less than n and prime to n. Therefore $\phi(n)$ positive integers less than n and prime to n can be expressed as $a_1, a_2, a_3, \dots, (n - a_3), (n - a_2), (n - a_1)$.

Let $S = a_1 + a_2 + a_3 + \dots + (n - a_3) + (n - a_2) + (n - a_1)$.

Then $S = (n - a_1) + (n - a_2) + (n - a_3) + \dots + a_3 + a_2 + a_1$. (sum in the reverse order)

Adding, we have $2S = \phi(n).n$, or, $S = \frac{1}{2}n.\phi(n)$.

2. For any positive integer n, $n = \sum_{d|n} \phi(d)$, where the summation extends over all positive divisors d of n.

Proof. Let $S = \{1, 2, \dots, n\}$, $n > 1$. Let $d_1, d_2, \dots, d_k$ be positive divisors of n. For a divisor d of n, let us define a subset S_d of S by

$$S_d = \{a \in S : gcd(a, n) = d\}.$$

Then each element of S belongs to exactly one of the subsets $S_{d_1}, S_{d_2}, \dots, S_{d_k}$ and $S = S_{d_1} \cup S_{d_2} \cup \dots \cup S_{d_k}$.

Let $d < n$. $a \in S_d \Rightarrow gcd(a, n) = d \Rightarrow a = \lambda d$ and $n = d'd$, where λ is less than $d'(= \frac{n}{d})$ and prime to d'. ∃ λ and d' prime to each other

Thus S_d contains integers of the type λd, where λ is less than d' and prime to d'. Therefore S_d contains $\phi(d')$ elements.

Since the total number of elements in S is n and they are distributed into distinct and disjoint subsets of S, $n = \sum_{d|n} \phi(d')$, where $dd' = n$.

As d runs through all positive divisors of n, d' does so.

Therefore $\sum_{d|n} \phi(d') = \sum_{d'|n} \phi(d') = \sum_{d|n} \phi(d)$. If $n = 1$, $\sum_{d|n} \phi(d) = \phi(1) = 1$.

This completes the proof.

To elucidate the proof, let us take $n = 15$. The divisors of 15 are 1,3,5 and 15.

$S_1 = \{a : gcd(a, 15) = 1\} = \{1, 2, 4, 7, 8, 11, 13, 14\}$;
$S_3 = \{a : gcd(a, 15) = 3\} = \{3, 6, 9, 12\}$;
$S_5 = \{a : gcd(a, 15) = 5\} = \{5, 10\}$;
$S_{15} = \{a : gcd(a, 15) = 15\} = \{15\}$.

S_1 contains $\phi(15)$ elements, S_3 contains $\phi(5)$ elements, S_5 contains $\phi(3)$ elements and S_{15} contains $\phi(1)$ elements.

$\phi(1) = 1, \phi(3) = 2, \phi(5) = 4, \phi(15) = 8$.

$\sum_{d|15} \phi(d) = \phi(1) + \phi(3) + \phi(5) + \phi(15) = 1 + 2 + 4 + 8 = 15$.

3. If $n > 2$ then $\phi(n)$ is an even integer.

Proof. **Case 1.** Let n be a power of 2. Let $n = 2^k$, $k > 1$.
Then $\phi(n) = \phi(2^k) = 2^k(1 - \frac{1}{2}) = 2^{k-1}$, an even integer.

Case 2. Let n be not a power of 2. Then n has an odd prime divisor, say, p. Let $n = p^\alpha q$, where $\alpha \geq 1$ and p, q are prime to each other.

Then $\phi(n) = \phi(p^\alpha).\phi(q)$, since p^α, q are prime to each other
$= p^\alpha(1 - \frac{1}{p}).\phi(q) = p^{\alpha-1}(p-1).\phi(q)$.

Since $(p-1)$ is even, it follows that $\phi(n)$ is even.

This completes the proof.

Another proof of Euclid's theorem on infinitude of primes.

Let the number of primes be finite and $p_1, p_2, \ldots, p_m$ be the enumeration of all primes. Let $p = p_1 p_2 \ldots p_m$.

Let k be an integer satisfying $2 \leq k \leq p$. Then by the fundamental theorem, k is either a prime, i.e., one of $p_1, p_2, \ldots, p_m$, or k has a prime divisor, say q which is one of $p_1, p_2, \ldots, p_m$.

Thus in any case, $gcd(k, p) > 1$. Therefore every integer k satisfying $1 < k \leq p$ is not prime to p and this gives $\phi(p) = 1$.

But p is clearly an integer > 2 and by the property of the phi function, $\phi(n)$ is even for every integer $n > 2$. We arrive at a contradiction.

Therefore the number of primes is infinite and this completes the proof.

Worked Examples.

1. Find the number of integers less than n and prime to n, when $n = 324, 900$.

$324 = 2^2.3^4$. Therefore $\phi(324) = 324(1 - \frac{1}{2})(1 - \frac{1}{3}) = 108$.

$900 = 2^2.3^2.5^2$. Therefore $\phi(900) = 900(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = 240$.

2. If n be an odd positive integer, prove that $\phi(2n) = \phi(n)$.

Since n is odd, $gcd(2, n) = 1$.
Therefore $\phi(2n) = \phi(2)\phi(n) = \phi(n)$, since $\phi(2) = 1$.

3. If n be an even positive integer, prove that $\phi(2n) = 2\phi(n)$.

Let $n = 2^k.p$, where p is an odd positive integer.

Then $\phi(n) = \phi(2^k).\phi(p) = 2^k(1 - \frac{1}{2}).\phi(p) = 2^{k-1}\phi(p)$ and
$\phi(2n) = \phi(2^{k+1}p) = \phi(2^{k+1})\phi(p) = 2^k\phi(p)$. Therefore $\phi(2n) = 2\phi(n)$.

Theorem 3.5.4. Fermat's theorem.

If p be a prime and p is not a divisor of a, then $a^{p-1} \equiv 1 \pmod{p}$.

Proof. Let us consider the integers $a, 2a, 3a, \ldots, (p-1)a$.

None of these is divisible by p. Also no two of these are congruent modulo p, because if $ra \equiv sa \pmod{p}$ for some integers r, s satisfying $1 \leq r < s \leq p-1$, then cancelling a (since a is prime to p) we must have $r \equiv s \pmod{p}$, a contradiction.

This means that the integers $a, 2a, 3a, \ldots, (p-1)a$ are congruent to $1, 2, 3, \ldots, p-1$ modulo p, taken in some order.

Therefore $a.2a.3a.\ldots.(p-1)a \equiv 1.2.3.\ldots.(p-1) \pmod{p}$,

or, $a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}$.

Since $gcd(p, (p-1)!) = 1$, this is equivalent to $a^{p-1} \equiv 1 \pmod{p}$.

This completes the proof.

Note. This theorem is popularly known as "Fermat's Little Theorem" to distinguish it from Fermat's "Great Theorem" or "Last Theorem", which states that "if $n > 2$, then the Diophantine equation $x^n + y^n = z^n$ has no solution in integers, other than the trivial solution in which at least one of the variables is zero."

Note. The theorem may be stated in a more general form –

If p be a prime and a is any integer, $a^p \equiv a \pmod{p}$.

When p is not a divisor of a, $a^{p-1} \equiv 1 \pmod{p}$ and in this case $a^p \equiv a \pmod{p}$ holds. And when p is a divisor of a, $a^p \equiv 0 \pmod{p}$ and it can be expressed as $a^p \equiv a \pmod{p}$.

Corollary. Since p is a prime, $p-1$ is an even integer except when $p = 2$. Therefore $(a^{\frac{p-1}{2}} + 1)(a^{\frac{p-1}{2}} - 1) \equiv 0 \pmod{p}$, when p is an odd prime and a is not divisible by p.

This implies $p \mid (a^{\frac{p-1}{2}} + 1)(a^{\frac{p-1}{2}} - 1)$.

Since p is a prime, either $p \mid (a^{\frac{p-1}{2}} + 1)$, or $p \mid (a^{\frac{p-1}{2}} - 1)$, but not both. Because $p \mid (a^{\frac{p-1}{2}} + 1)$ and $p \mid (a^{\frac{p-1}{2}} - 1)$ together implies $p \mid 2$, an impossibility.

Therefore $a^{\frac{p-1}{2}}$ is either $pk + 1$ or $pk - 1$, where k is some integer.

As an application of the corollary, let us take $p = 7$. Then for any integer a not divisible by 7, a^3 is either $7k + 1$ or $7k - 1$.

When a is divisible by 7, $a^3 = 7k$.

It follows that for any integer a, a^3 is one of the forms $7k, 7k \pm 1$, where k is an integer.

Theorem 3.5.5. Euler's theorem.

If n be a positive integer and a is prime to n, then $a^{\phi(n)} \equiv 1 \pmod{n}$.

Proof. The theorem holds trivially if $n = 1$. Let $n > 1$ and let $a_1, a_2, \ldots, a_{\phi(n)}$ be positive integers less than n and prime to n.

Let us consider the integers $aa_1, aa_2, \ldots, aa_{\phi(n)}$. Each of these is prime to n, since $gcd(a, n) = 1$ and $gcd(a_i, n) = 1 \Rightarrow gcd(aa_i, n) = 1$.

No two of the integers $aa_1, aa_2, \ldots, aa_{\phi(n)}$ are congruent modulo n, because $aa_i \equiv aa_j \pmod{n} \Rightarrow a_i \equiv a_j \pmod{n}$, since $gcd(a, n) = 1$, but $a_i \not\equiv a_j \pmod{n}$.

Therefore the integers $aa_1, aa_2, \ldots, aa_{\phi(n)}$ contain exactly $\phi(n)$ elements and they are congruent to the integers $a_1, a_2, \ldots, a_{\phi(n)}$ modulo n in some order.

Multiplying the $\phi(n)$ congruences, we have

$aa_1.aa_2 \ldots aa_{\phi(n)} \equiv a_1.a_2 \ldots a_{\phi(n)} \pmod{n}$.

Since $gcd(a_i, n) = 1$ for each i, it follows that $gcd(a_1 a_2 \ldots a_{\phi(n)}, n) = 1$. Therefore we can cancel $a_1 a_2 \ldots a_{\phi(n)}$ from both sides of the congruence and obtain $a^{\phi(n)} \equiv 1 \pmod{n}$.

This completes the proof.

Note 1. Euler's theorem is a generalisation of Fermat's theorem.

Euler's theorem holds for any integer n. If, in particular, n be a prime, then $\phi(n) = n - 1$ and in that case, for any integer a prime to n, Euler's theorem gives $a^{n-1} \equiv 1 \pmod{n}$.

Note 2. If m be a positive integer and $gcd(a, m) = 1$, the linear congruence $ax \equiv 1 \pmod{m}$ has a unique solution $(\text{mod } m)$. Euler's theorem suggests a method of finding such a solution. Since $a^{\phi(m)} \equiv 1 \pmod{m}$, the unique solution is given by $x \equiv a^{\phi(m)-1} \pmod{m}$.

Theorem 3.5.6. Wilson's theorem.

If p be a prime then $(p-1)! + 1 \equiv 0 (\text{mod } p)$.

Proof. Let a be one of the integers $1, 2, 3, \ldots, p-1$. Then no two of the integers $1.a, 2.a, 3.a, \ldots, (p-1)a$ are congruent modulo p, because if $ra \equiv sa \pmod{p}$ for some integers r, s satisfying $1 \leq r < s \leq p-1$, then cancelling a (since a is prime to p) we must have $r \equiv s (\text{mod } p)$, a contradiction. Also none of these is divisible by p.

This means that the integers $a, 2a, 3a, \ldots, (p-1)a$ are congruent to $1, 2, 3, \ldots, p-1$ modulo p, taken in some order. Hence for every a, there

is one a', and only one, such that $a'a \equiv 1 \pmod{p}$.

$a^2 \equiv 1 \pmod{p}$ holds if $p \mid (a^2 - 1)$ and this happens only when $p \mid (a-1)$ or $p \mid (a+1)$. Since p is a prime and $a < p$, it follows that in this case either $a = 1$ or $a = p - 1$.

Therefore 1 and $p-1$ are the only values of a for which $a.a \equiv 1 \pmod{p}$.

If we omit integers 1 and $p-1$, the remaining integers $2, 3, \ldots, p-2$ are such that they are grouped into pairs a, a' for which $a.a' \equiv 1 \pmod{p}$ holds. Multiplying $\frac{1}{2}(p-3)$ pairs of such congruences, we have

$$2.3 \ldots (p-2) \equiv 1 \pmod{p}$$

or, $$(p-2)! \equiv 1 \pmod{p}.$$

Multiplying by $p-1$, $(p-1)! \equiv p-1 \pmod{p} \equiv -1 \pmod{p}$ which is equivalent to $(p-1)! + 1 \equiv 0 \pmod{p}$.

This completes the proof.

Let us take an example to elucidate the proof of the theorem.

Let $p = 11$. Then the integers $1, 2, 3, \ldots, 10$ are grouped into pairs a, a' such that $a.a' \equiv 1 \pmod{11}$.

$1.1 \equiv 1 \pmod{11}$, $2.6 = 1 \pmod{11}$, $3.4 \equiv 1 \pmod{11}$,
$5.9 \equiv 1 \pmod{11}$, $7.8 \equiv 1 \pmod{11}$, $10.10 \equiv 1 \pmod{11}$.

Omitting the first and the last and multiplying the remaining,

$$9! \equiv 1 \pmod{11}$$

or, $$10! \equiv 10 \pmod{11} \equiv -1 \pmod{11}$$

or, $$10! + 1 \equiv 0 \pmod{11}.$$

Note 1. The converse of Wilson's theorem is also true.

If $(p-1)! + 1 \equiv 0 \pmod{p}$ then p must be prime. For, if p be not a prime, then p has a divisor d with $1 < d < p$ such that d divides $(p-1)! + 1$. Again since $1 < d < p$, d divides one of the factors in $(p-1)!$. Therefore d divides both $(p-1)! + 1$ and $(p-1)!$ which yields $d \mid 1$, an absurdity.

Note 2. Wilson's theorem and its converse, taken together, provide a necessary and sufficient condition for determining primality of a positive integer n. But this test is impracticable since $(n-1)!$ becomes very large as n assumes large values.

We now come to a theorem of Dirichlet that establishes the existence of infinity of primes in the arithmetic progression $1 + 4n, n \in \mathbf{N}$.

Theorem 3.5.7. (Dirichlet) There are infinitely many primes of the form $4n+1$.

Proof. Let N be any positive integer. We shall show that there exists a prime $> N$ of the form $4n+1$.

Let $m = (N!)^2 + 1$. Then m is odd and $m > 1$. None of the numbers $2, 3, \ldots, N$ is a divisor of m. Let p be the smallest prime divisor of m. Then $p > N$.

We have $(N!)^2 \equiv -1 (\text{mod } p)$.

Therefore $(N!)^{p-1} \equiv (-1)^{\frac{p-1}{2}} (\text{mod } p)$.

But $(N!)^{p-1} \equiv 1 (\text{mod } p)$, by Fermat's theorem.

Hence $(-1)^{\frac{p-1}{2}} \equiv 1 (\text{mod } p)$.

$(-1)^{\frac{p-1}{2}}$ is either 1 or -1. If $(-1)^{\frac{p-1}{2}} = -1$, then $-1 \equiv 1 (\text{mod } p)$ must hold. But it is an impossibility, since p is odd. Therefore $(-1)^{\frac{p-1}{2}}$ is 1 and therefore $\frac{p-1}{2}$ is even. That is, $p \equiv 1 (\text{mod } 4)$.

Thus for every positive integer N there exists a prime $p > N$ such that $p \equiv 1 (\text{mod } 4)$ and consequently, the number of primes of the form $4n+1$ is infinite.

This completes the proof.

Worked Examples.

1. Find the least positive residue in $2^{41} \pmod{23}$.

23 is a prime and 2 is prime to 23.

By Fermat's theorem, $2^{22} \equiv 1 \pmod{23}$.

Therefore $2^{44} \equiv 1 \pmod{23} \equiv 24 \pmod{23}$

or, $2^{41}.8 \equiv 3.8 \pmod{23}$.

Since 8 is prime to 23, we have $2^{41} = 3 \pmod{23}$.

Hence the least positive residue is 3.

2. If p be a prime > 2, prove that $1^p + 2^p + \cdots + (p-1)^p \equiv 0 \pmod{p}$.

By Fermat's theorem,

$1^p \equiv 1 \pmod{p}$, $2^p \equiv 2 \pmod{p}, \ldots, (p-1)^p \equiv p-1 \pmod{p}$.

Adding the congruences, we have

$$1^p + 2^p + \cdots + (p-1)^p \equiv \{1 + 2 + \cdots + (p-1)\} \pmod{p}$$
$$\equiv \frac{p(p-1)}{2} \pmod{p}$$
$$\equiv 0 \pmod{p} \text{ since } p-1 \text{ is even.}$$

3. Prove that the eighth power of any integer is of the form $17k$ or $17k \pm 1$.

If a be an integer divisible by 17 then $a^8 = 17k$.

If a is not divisible by 17 then a is prime to 17.

By Fermat's theorem, $a^{16} - 1 \equiv 0 \pmod{17}$

or, $(a^8 - 1)(a^8 + 1) \equiv 0 \pmod{17}$.

Either $a^8 - 1 \equiv 0 (\text{mod } 17)$, or $a^8 + 1 \equiv 0 (\text{mod } 17)$.

$a^8 - 1 \equiv 0 (\text{mod } 17) \Rightarrow a^8 = 17k + 1$

$a^8 + 1 \equiv 0 (\text{mod } 17) \Rightarrow a^8 = 17k - 1$.

Hence $a^8 = 17k$ or $17k \pm 1$ where a is an integer.

4. Show that $a^{12} - b^{12}$ is divisible by 91 if a and b are both prime to 91.

Since a is prime to 91, a is prime to both 13 and 7.

By Fermat's theorem,

$a^{12} - 1 \equiv 0 \pmod{13}$ and $a^6 - 1 \equiv 0 \pmod{7}$.

Since $a^6 - 1 \equiv 0 \pmod{7}$, it follows that $a^{12} - 1 \equiv 0 \pmod{7}$.

Now $a^{12} - 1 \equiv 0 \pmod{13}$ and $a^{12} - 1 \equiv 0 \pmod{7}$ together imply $a^{12} - 1 \equiv 0 \pmod{91}$, since 13 and 7 are prime to each other.

Similarly, $b^{12} - 1 \equiv 0 \pmod{91}$.

Therefore $a^{12} - b^{12} \equiv 0 \pmod{91}$.

5. If n is a prime > 7, prove that $n^6 - 1$ is divisible by 504.

Since 7 is a prime and n is prime to 7, $n^6 - 1$ is divisible by 7, by Fermat's theorem.

Since n is prime to 9, $n^{\phi(9)} - 1$ is divisible by 9, by Euler's theorem.

$\phi(9) = 9.(1 - \frac{1}{3}) = 6$. Therefore $n^6 - 1$ is divisible by 9.

Since n is an odd prime > 7, n is one of the forms $4k+1$ or $4k+3$, where k is an integer > 1.

$n^6 - 1 = (n-1)(n+1)(n^4 + n^2 + 1)$.

If $n = 4k+1$, then $(n-1)(n+1) = 4k(4k+2)$. It is divisible by 8.

If $n = 4k+3$, then $(n-1)(n+1) = (4k+2)(4k+4)$. It is divisible by 8.

Therefore in any case, $n^6 - 1$ is divisible by 8.

Since 7, 8, 9 are pairwise prime to each other, $n^6 - 1$ is divisible by 7.8.9, i.e., by 504.

6. Prove that $\frac{n^7}{7} + \frac{n^3}{3} + \frac{11n}{21}$ is an integer for all $n \in \mathbf{N}$.

We have $n^7 \equiv n \pmod 7$ and $n^3 \equiv n \pmod 3$ for all natural numbers n. Hence $n^7 - n$ is a multiple of 7 and $n^3 - n$ is a multiple of 3.

Therefore there exist integers r and t such that $n^7 - n = 7r$ and $n^3 - n = 3t$.

$$\frac{n^7}{7} + \frac{n^3}{3} + \frac{11n}{21} = \frac{7r+n}{7} + \frac{3t+n}{3} + \frac{11n}{21}$$
$$= (r+t) + \frac{n}{7} + \frac{n}{3} + \frac{11n}{21}$$
$$= (r+t) + \frac{3n+7n+11n}{21} = r + t + n, \text{ an integer.}$$

7. Prove that $abc(a^5 - b^5)(b^5 - c^5)(c^5 - a^5)$ is divisible by 11 for all integers a, b, c.

Case 1. Let one of a, b, c be divisible by 11.

Then abc is divisible by 11 and therefore $abc(a^5 - b^5)(b^5 - c^5)(c^5 - a^5)$ is divisible by 11

Case 2. Let none of a, b, c be divisible by 11.

By Fermat's theorem, $a^{10} - 1 \equiv 0 \pmod{11}$. Then $11 | (a^5 - 1)(a^5 + 1)$.

Since 11 is a prime, either $11 | (a^5 - 1)$ or $11 | (a^5 + 1)$.

Therefore a^5 when divided by 11, leaves remainder 1 or 10.

Since the fifth power of any integer, not divisible by 11, leaves only one of the two possible remainders 1 or 10 when divided by 11, at least two of a^5, b^5, c^5 must leave the same remainder when divided by 11.

Therefore $(a^5 - b^5)(b^5 - c^5)(c^5 - a^5)$ is divisible by 11 and therefore $abc(a^5 - b^5)(b^5 - c^5)(c^5 - a^5)$ is divisible by 11.

8. If p and q are distinct primes, prove that $p^{q-1} + q^{p-1} - 1$ is divisible by pq.

Since p is a prime and q is prime to p, $q^{p-1} - 1$ is divisible by p, by Fermat's theorem.

Since $q > 1$, p^{q-1} is divisible by p.

Therefore $p^{q-1} + q^{p-1} - 1$ is divisible by p.

Since q is a prime and p is prime to q, $p^{q-1} - 1$ is divisible by q, by Fermat's theorem.

Since $p > 1$, q^{p-1} is divisible by q.

Therefore $p^{q-1} + q^{p-1} - 1$ is divisible by q.

Since p and q are prime to each other, it follows that $p^{q-1} + q^{p-1} - 1$ is divisible by pq.

9. Prove that no prime factor of $n^2 + 1$ can be of the form $4m - 1$, where m is an integer.

Let p be a prime factor of $n^2 + 1$. Then p is not a divisor of n and $n^2 + 1 \equiv 0 \pmod p$.

By Fermat's theorem, $n^{p-1} \equiv 1 \pmod p$.

Let us assume that p is odd.

Then $(n^2)^{\frac{p-1}{2}} \equiv 1 \pmod p$ ... (i)

Again $n^2 \equiv -1 \pmod p \Rightarrow (n^2)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} \pmod p$ ... (ii)

From (i) and (ii), we have $1 \equiv (-1)^{\frac{p-1}{2}} \pmod p$.

Hence $\frac{p-1}{2}$ is even and therefore $p = 4k + 1$, where k is an integer.

Therefore p is not of the form $4m - 1$. If however, p be an even prime, then p is also not of the form $4m - 1$.

10. If p be a prime and a is prime to p, prove that $a^{p^2-p} \equiv 1 \pmod{p^2}$.

By Fermat's theorem, $a^{p-1} \equiv 1 \pmod p$.

Therefore $a^{p-1} = 1 + tp$ for some integer t.

Hence $a^{p^2-p} = (1 + tp)^p$
$= 1 + p.tp + p_{C_2}.(t^2p^2) + \cdots + (tp)^p$
$= 1 + kp^2$ where k is an integer .

Consequently, $a^{p^2-p} \equiv 1 \pmod{p^2}$.

11. Use Euler's theorem to find the units digit in 3^{100}.

Since 3 is prime to 10, $3^{\phi(10)} \equiv 1 \pmod{10}$, by Euler's theorem.

$\phi(10) = 4$. Therefore $3^4 \equiv 1 \pmod{10}$.

$3^{100} = 3^{4.25} \equiv 1 \pmod{10}$.

Consequently, the units digit in 3^{100} is 1.

12. Show that $4(29)! + 5!$ is divisible by 31.

By Wilson's theorem, $(30)! + 1 \equiv 0 \pmod{31}$, since 31 is a prime.

or, $(31 - 1)(29)! + 1 \equiv 0 \pmod{31}$

or, $-(29)! + 1 \equiv 0 \pmod{31}$

or, $(29)! - 1 \equiv 0 \pmod{31}$

or, $4(29)! - 4 \equiv 0 \pmod{31}$

or, $4(29)! - 4 + 124 \equiv 0 \pmod{31}$

or, $4(29)! + 120 \equiv 0 \pmod{31}$

or, $4(29)! + 5! \equiv 0 \pmod{31}$.

Therefore $4(29)! + 5!$ is divisible by 31.

13. If $2n+1$ is prime, prove that $(n!)^2 \equiv (-1)^{n+1} \pmod{(2n+1)}$.

By Wilson's theorem, $(2n)! \equiv -1 \pmod{2n+1}$.

$(2n)! \equiv n!(n+1)(n+2)\ldots 2n$.

$$\begin{aligned} n+1 &\equiv -n \pmod{2n+1} \\ n+2 &\equiv -(n-1) \pmod{2n+1} \\ \ldots\quad &\ldots\quad \ldots \\ 2n &\equiv -1 \pmod{2n+1}. \end{aligned}$$

Therefore $(n+1)(n+2)\ldots 2n \equiv (-1)^n n! \pmod{2n+1}$.

$$\begin{aligned} \text{Hence } (2n)! &\equiv (-1)^n (n!)^2 \pmod{2n+1} \\ \text{or } (n!)^2 &\equiv (-1)^n (2n)! \pmod{2n+1} \\ &\equiv (-1)^n .-1 \pmod{2n+1} \\ &\equiv (-1)^{n+1} \pmod{2n+1}. \end{aligned}$$

Exercises 3C

1. Find $\phi(n)$, where

(i) $n = 2048$, (ii) $n = 5040$, (iii) $n = 7200$.

2. If a positive integer n has r distinct odd prime divisors, prove that 2^r is a divisor of $\phi(n)$.

3. Prove that $\phi(3n) = 3\phi(n)$ if and only if 3 is a divisor of n.

4. Prove that $\phi(n^2) = n\phi(n)$, for every positive integer n.

5. Prove that $\phi(n) = \frac{n}{2}$ if and only if $n = 2^k$ for some integer $k \geq 1$.

6. m, n are positive integers and $gcd(m,n) = d$. Prove that $\phi(mn) = \phi(m)\phi(n)\frac{d}{\phi(d)}$.

7. If every prime that divides n also divides m, show that $\phi(mn) = n\phi(m)$.

8. If the same primes divide m and n, prove that $n\phi(m) = m\phi(n)$.

9. If m, n be positive integers, prove that $\phi(mn) = \phi((m,n))\phi([m,n])$, where (m,n) = the g.c.d. of a and b and $[m,n]$ = the l.c.m. of a and b.

10. If n be the product of a pair of twin primes, prove that $\phi(n)\sigma(n) = (n+1)(n-3)$.

[A pair of successive odd integers both of which are primes, is said to be a *twin prime*.]

[**Hint.** Let $n = p(p+2)$, where p and $p+2$ are primes. $\phi(n) = \phi(p).\phi(p+2) = (p-1)(p+1), \sigma(n) = \sigma(p).\sigma(p+2) = (p+1)(p+3)$.]

11. (i) Prove that the 5th power of any integer is of the form $11k$ or $11k \pm 1$.

(ii) Prove that the 9th power of any integer is of the form $19k$ or $19k \pm 1$.

12. Prove that

(i) $\frac{5}{11}n^{11} + \frac{1}{3}n^3 + \frac{19}{33}n$ is an integer for every integer n;

(ii) $\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is an integer for every integer n.

13. Prove that $abc(a^3 - b^3)(b^3 - c^3)(c^3 - a^3)$ is divisible by 7 for all integers a, b, c.

14. Show that

(i) $a^{18} - b^{18}$ is divisible by 133 if a and b are both prime to 133.

(ii) $a^{16} - b^{16}$ is divisible by 133 if a and b are both prime to 85.

15. If m and n are relatively prime integers, prove that $m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}$.

16. If p and q are distinct primes and a is any integer, prove that $a^{pq} - a^p - a^q + a$ is divisible by pq.

17. Find the remainder when

(i) 333^{333} is divided by 7, (ii) 777^{777} is divided by 16.

18. If p and $p+2$ be a pair of twin primes, prove that $4(p-1)! + p + 4 \equiv 0 \pmod{p(p+2)}$.

[**Hint.** By Wilson's theorem, $(p-1)! + 1 \equiv 0 \pmod{p}$ (i) and $(p+1)! + 1 \equiv 0 \pmod{p+2}$ (ii). From (i) $4(p-1)! + p + 4 \equiv 0 \pmod{p}$. From (ii) $[p(p+2) - (p+2) + 2](p-1)! + 1 \equiv 0 \pmod{p+2}$. Deduce that $4(p-1)! + p + 4 \equiv 0 \pmod{p+2}$.]

19. Find the units digit in (i) 7^{99}, (ii) 77^{77}.

20. Find the last two digits in (i) 7^{100}, (ii) 33^{100}.

[**Hint.** (i) By Euler's theorem, $7^{\phi(100)} \equiv 1 \pmod{100}$.]

21. If p is an odd prime, prove that

(i) $1^2.3^2.5^2\ldots(p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}$

(ii) $2^2.4^2.6^2\ldots(p-1)^2 \equiv (-1)^{(p+1)/2} \pmod{p}$.

[**Hint.** Use $k \equiv -(p-k) \pmod{p}$. Apply Wilson's theorem.]

22. If p is a prime, prove that $2(p-3)! + 1 \equiv 0 \pmod{p}$.

[**Hint.** $(p-1)! + 1 = (p^2 - 3p + 2)[(p-3)!] + 1 = p(p-3)[(p-3)!] + 2(p-3)! + 1$. Apply Wilson's theorem.]

3.6. Greatest integer function.

For any real number x, there exists an integer n such that $n \le x < n+1$, i.e., either x is an integer n or x lies between two consecutive integers n and $n+1$. Therefore x can be expressed as

$$x = n + f, \text{ where } n \text{ is an integer and } 0 \le f < 1.$$

The integer n is said to be the *integral part* of x and is denoted by $[x]$. f is said to be the *fractional part* of x and $f = x - [x]$.

So we have the relation $[x] \le x < [x] + 1$ for any real number x. As every real number x corresponds to an integer $[x]$ which is the greatest integer not greater than x by the correspondence $[\ ]$, $[\ ]$ is said to be the *greatest integer function.*

For example, $[.3] = 0$, $[3] = 3$, $[-.3] = -1$, $[\pi] = 3$, $[\sqrt{2}] = 1$.

Properties.

1. $[a+b] \ge [a] + [b]$ for all real numbers a, b.

Proof. Let $a = n_1 + f_1$, $b = n_2 + f_2$ where n_1, n_2 are integers and $0 \le f_1 < 1$, $0 \le f_2 < 1$.
Then $[a] = n_1$, $[b] = n_2$, $0 \le f_1 + f_2 < 2$.

$$[a+b] = [(n_1+n_2) + (f_1+f_2)] = n_1 + n_2, \text{ if } 0 \le f_1 + f_2 < 1$$
$$= n_1 + n_2 + 1, \text{ if } 1 \le f_1 + f_2 < 2.$$

Therefore $[a+b] \ge [a] + [b]$.

Corollary. $[a+b+c] \ge [a+b] + [c] \ge [a] + [b] + [c]$ for all real a, b, c.

In general, $[a_1 + a_2 + \cdots + a_n] \ge [a_1] + [a_2] + \cdots + [a_n]$ for all real numbers $a_1, a_2, \ldots, a_n$.

2. $[a] + [-a] = 0$, if a is an integer
$= -1$, otherwise.

Proof. **Case 1.** a is an integer.

Then $-a$ is an integer. $[a] = a, [-a] = -a$.
Therefore $[a] + [-a] = 0$.

Case 2. a is not an integer.

Let $a = n + f$, where n is an integer and $0 < f < 1$.
Then $[a] = n$. $0 < f < 1 \Rightarrow -1 < -f < 0 \Rightarrow 0 < 1 - f < 1$.

$-a = -n - f = -n - 1 + (1 - f)$ and therefore $[-a] = -n - 1$.
Consequently, $[a] + [-a] = n + (-n) = 0$.

The proof is complete.

3. $[a] + [a + \frac{1}{2}] = [2a]$ for all real a.

Proof. Let $a = n + f$, where n is an integer and $0 \le f < 1$.

Then $[a] = n$, $2a = 2n + 2f$. $0 \le f < 1 \Rightarrow 0 \le 2f < 2$.

Case 1. Let $0 \le 2f < 1$. Then $0 \le f < \frac{1}{2}$ and $\frac{1}{2} \le f + \frac{1}{2} < 1$.
Therefore $[a + \frac{1}{2}] = [n + f + \frac{1}{2}] = n$ and $[2a] = [2n + 2f] = 2n$.

Thus $[a] + [a + \frac{1}{2}] = n + n = 2n = [2a]$.

Case 2. Let $1 \le 2f < 2$. Then $\frac{1}{2} \le f < 1$ and $1 \le f + \frac{1}{2} < 1 + \frac{1}{2}$.
Therefore $[a + \frac{1}{2}] = [n + f + \frac{1}{2}] = n + 1$ and $[2a] = [2n + 2f] = 2n + 1$.

Thus $[a] + [a + \frac{1}{2}] = n + (n + 1) = 2n + 1 = [2a]$.

This completes the proof.

Theorem 3.6.1. The largest exponent e of a prime p such that p^e is a divisor of $n!$ is given by

$$e = [\tfrac{n}{p}] + [\tfrac{n}{p^2}] + [\tfrac{n}{p^3}] + \cdots$$

Proof. Let r be the least exponent of p such that $p^r > n$. Then $[\frac{n}{p^r}] = 0$ and this indicates that the series representing e is not really an infinite series.

If $p, 2p, 3p, \ldots, mp$ are all the positive integral multiples of p not exceeding n, then $mp \le n < (m+1)p$. Therefore $m \le \frac{n}{p} < m + 1$. This implies $[\frac{n}{p}] = m$. This means that there are $[\frac{n}{p}]$ integers $\le n$ that are divisible by p.

Out of these $[\frac{n}{p}]$ integers some may be divisible by p again and they are $p^2, 2p^2, 3p^2, \ldots$. The number of such integers is $[\frac{n}{p^2}]$.

Out of these $[\frac{n}{p^2}]$ integers some may be divisible by p again and the number of such integers is $[\frac{n}{p^3}]$.

This process is continued till we obtain some exponent r such that $p^r > n$ and in that case $[\frac{n}{p^r}] = 0$.

Hence the total number of times p divides $n!$ is

$$[\tfrac{n}{p}] + [\tfrac{n}{p^2}] + [\tfrac{n}{p^3}] + \cdots$$

To illustrate the theorem, let $n = 60$, $p = 3$.

$60! = 1.2.3....60$. The integers ≤ 60 and divisible by 3 are $3, 2.3, 3.3, \ldots, 20.3$. The number of such integers is $[\frac{60}{3}]$.

Some of these $[\frac{60}{3}] (= 20)$ integers are divisible by 3^2. They are $3^2, 2.3^2, \ldots, 6.3^2$. The number of such integers is $[\frac{20}{3}]$, *i.e.*, $[\frac{60}{3^2}]$.

Some of these $[\frac{60}{3^2}](=6)$ integers are divisible by 3^3. They are $3^3, 2.3^3$. The number of such integers is $[\frac{6}{3}]$, *i.e.*, $[\frac{60}{3^3}]$.

None of these $[\frac{60}{3^3}](=2)$ integers is divisible by 3^4.

The process terminates.

Therefore the number of times the prime divisor 3 is repeated in the product 1.2.3....60 is $[\frac{60}{3}]+[\frac{60}{3^2}]+[\frac{60}{3^3}] = 20+6+2 = 28$.

Theorem 3.6.2. The product of any n consecutive positive integers is divisible by $n!$.

Proof. Let the first factor in the product be $m+1$. Then the product $= (m+1)(m+2)\dots(m+n) = \frac{(m+n)!}{m!}$.

The theorem will be proved if we can prove that $\frac{(m+n)!}{m!n!}$ is an integer.

Let p be any prime divisor of the denominator $m!n!$. Then clearly, p is also a divisor of the numerator $(m+n)!$.

The largest exponent r of p such that p^r is a divisor of $(m+n)!$ is given by $r = [\frac{m+n}{p}]+[\frac{m+n}{p^2}]+[\frac{m+n}{p^3}]+\cdots$

The largest exponent s of p such that p^s is a divisor of $m!$ is given by $s = [\frac{m}{p}]+[\frac{m}{p^2}]+[\frac{m}{p^3}]+\cdots$.

The largest exponent t of p such that p^t is a divisor of $n!$ is given by $t = [\frac{n}{p}]+[\frac{n}{p^2}]+[\frac{n}{p^3}]+\cdots$

But $[\frac{m+n}{p}] \geq [\frac{m}{p}]+[\frac{n}{p}]$, $[\frac{m+n}{p^2}] \geq [\frac{m}{p^2}]+[\frac{n}{p^2}]$, ..., by the property 1.

Therefore $r \geq s+t$. This implies that every prime divisor p of $m!n!$ appears in the numerator of the fraction $\frac{(m+n)!}{m!n!}$ at least as many times as it occurs in the denominator. Therefore $m!n!$ must be a divisor of $(m+n)!$ and the fraction $\frac{(m+n)!}{m!n!}$ turns out to be an integer.

This completes the proof.

Theorem 3.6.3. If n be a prime then ${}^nC_r$ is divisible by n for $0 < r < n$.

Proof. ${}^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$. The numerator being the product of r consecutive factors, is divisible by $r!$, by the previous theorem. Therefore $r!$ is a divisor of $n(n-1)(n-2)\dots(n-r+1)$.

Since n is a prime and r is less than n, $r!$ is prime to n.

$r!|n(n-1)(n-2)\dots(n-r+1)$ and $gcd(r!, n) = 1$ together imply that $r!|(n-1)(n-2)\dots(n-r+1)$. Therefore $\frac{(n-1)(n-2)\dots(n-r+1)}{r!}$ is an integer, k say.

Consequently, $\frac{n(n-1)(n-2)\dots(n-r+1)}{r!} = nk$ and this proves that ${}^nC_r$ is divisible by n.

This completes the proof.

Another proof of Fermat's theorem.

If p be a prime and p is not a divisor of n, $n^{p-1} \equiv 1 \pmod p$.

Proof. First we prove that if p be a prime, $n^p \equiv n \pmod p$ for all positive integers n, by using the principle of induction.

The statement $n^p \equiv n \pmod p$ is trivially true for $n = 1$.

Let us assume that the statement is true for $n = k$, $k \geq 1$.

Then $k^p \equiv k \pmod p$.

Now $(k+1)^p = k^p + {}^pc_1 k^{p-1} + {}^pc_2 k^{p-2} + \cdots + {}^pc_{p-1}k + 1$

$= k^p + M(p) + 1$, where $M(p) = {}^pc_1 k^{p-1} + {}^pc_2 k^{p-2} + \cdots + {}^pc_{p-1}k$.

$M(p)$ is divisible by p, since ${}^pc_r$ is divisible by p for $0 < r < p$.

Therefore $(k+1)^p \equiv k+1 \pmod p$ and this shows that the statement $n^p \equiv n \pmod p$ is true for $n = k+1$, if it be true for $n = k$. By the principle of induction, the statement $n^p \equiv n \pmod p$ is true for all natural numbers n.

Case 1. n is a positive integer prime to p.

Cancelling n from both sides of the congruence $n^p \equiv n \pmod p$, we have $n^{p-1} \equiv 1 \pmod p$.

Case 2. n is a negative integer prime to p.

There exist integers q and r such that $n = pq + r$, where $0 < r < p$.

Then $n^{p-1} \equiv r^{p-1} \pmod p$, where r is a positive integer $< p$

$\equiv 1 \pmod p$, by case 1.

This completes the proof.

Worked Examples.

1. Find the highest power of 5 contained in 140!.

$[\frac{140}{5}] = 28$, $[\frac{140}{5^2}] = [\frac{28}{5}] = 5$, $[\frac{140}{5^3}] = [\frac{5}{5}] = 1$.

Therefore the required power is $28+5+1 = 34$.

2. Find the number of zeros at the right end of the integer 141!.

We are to find the greatest exponent of 10 contained in the integer 141!.

The greatest exponent of 2 contained in the integer $141! = [\frac{141}{2}]+[\frac{141}{2^2}]+[\frac{141}{2^3}]+\dots$

$[\frac{141}{2}] = 70$, $[\frac{141}{2^2}] = [\frac{70}{2}] = 35$, $[\frac{141}{2^3}] = [\frac{35}{2}] = 17$, $[\frac{141}{2^4}] = [\frac{17}{2}] = 8$, $[\frac{141}{2^5}] = [\frac{8}{2}] = 4$, $[\frac{141}{2^6}] = [\frac{4}{2}] = 2$, $[\frac{141}{2^7}] = [\frac{2}{2}] = 1$, $[\frac{141}{2^8}] = [\frac{1}{2}] = 0$.

The greatest exponent of 2 contained in the integer $141! = 70 + 35 + 17 + 8 + 4 + 2 + 1 = 137$.

The greatest exponent of 5 contained in the integer $141! = [\frac{141}{5}] + [\frac{141}{5^2}] + [\frac{141}{5^3}] + \cdots$

$[\frac{141}{5}] = 28$, $[\frac{141}{5^2}] = [\frac{28}{5}] = 5$, $[\frac{141}{5^3}] = [\frac{5}{5}] = 1$, $[\frac{141}{5^4}] = [\frac{1}{5}] = 0$.

The greatest exponent of 5 contained in the integer $141! = 28 + 5 + 1 = 34$.

The greatest exponent of 10 contained in the integer 141! is the minimum of 67 and 34, i.e., 34.

Hence the number of zeros at the right end of the integer 141! is 34.

3.7. Mobius function.

Definition. The *Mobius function* μ is defined for all positive integers by

$\mu(1) = 1$;

$\mu(n) = (-1)^k$ if $n = p_1 p_2 \ldots p_k$, where $p_1, p_2, \ldots, p_k$ are distinct primes;

$\mu(n) = 0$ if $p^2 | n$ for some prime p.

To be explicit, $\mu(n) = 0$ if n is not square-free and $\mu(n) = (-1)^k$ if n is square free and is the product of k distinct primes.

For example, $\mu(1) = 1$, $\mu(2) = -1$, $\mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$, $\mu(6) = 1$, $\mu(7) = -1$, $\mu(8) = 0$, $\mu(9) = 0$, $\mu(10) = 1$.

Theorem 3.7.1. The function μ is a multiplicative function. That is, if m and n be relatively prime positive integers, then $\mu(mn) = \mu(m)\mu(n)$.

Proof. **Case 1.** Either $m = 1$ or $n = 1$.

If $m = 1$, $\mu(mn) = \mu(n) = \mu(m)\mu(n)$, since $\mu(m) = 1$.

If $n = 1$, $\mu(mn) = \mu(m) = \mu(m)\mu(n)$, since $\mu(n) = 1$.

Case 2. Either m or n has a prime square divisor.

In this case the integer mn has a prime square divisor. $\mu(mn) = 0$.

If m has a prime square divisor, $\mu(m) = 0$. Therefore $\mu(mn) = \mu(m)\mu(n)$.

If n has a prime square divisor, $\mu(n) = 0$. Therefore $\mu(mn) = \mu(m)\mu(n)$.

Case 3. m, n are both square-free.

Let $m = p_1 p_2 \ldots p_k$, $n = q_1 q_2 \ldots q_r$, where p_i, q_j are distinct primes.

$\mu(m) = (-1)^k$. $\mu(n) = (-1)^r$.

Then $mn = p_1 p_2 \ldots p_k q_1 q_2 \ldots q_r$. $\mu(mn) = (-1)^{k+r}$.

Therefore $\mu(mn) = \mu(m)\mu(n)$ holds.

This completes the proof.

Theorem 3.7.2. If n be a positive integer, $\sum_{d|n} \mu(d) = 1$ if $n = 1$

$= 0$ if $n > 1$.

Proof. **Case 1.** $n = 1$. In this case $d = 1$ and $\sum_{d|n} \mu(d) = \mu(1) = 1$.

Case 2. $n > 1$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_i \geq 1$.

In the sum $\sum_{d|n} \mu(d)$, the only non-zero terms come from the divisor $d = 1$ and from those divisors d which are primes or the product of non-repeated primes.

Thus $\sum_{d|n} \mu(d) = \mu(1) + [\mu(p_1) + \mu(p_2) + \cdots + \mu(p_k)] + [\mu(p_1 p_2) + \mu(p_1 p_3) + \cdots + \mu(p_{k-1} p_k)] + \cdots + \mu(p_1 p_2 \ldots p_k)$.

In the group $[\mu(p_1) + \mu(p_2) + \cdots + \mu(p_k)]$, there are ${}^k c_1$ terms and each term is -1.

In the group $[\mu(p_1 p_2) + \mu(p_1 p_3) + \cdots + \mu(p_{k-1} p_k)]$, there are ${}^k c_2$ terms and each term is $(-1)^2$.

... ...

... ...

As $\mu(p_1 p_2 \ldots p_k) = (-1)^k$, the last group containing only one term can be exhibited as ${}^k c_k (-1)^k$.

Therefore $\sum_{d|n} \mu(d) = 1 + {}^k c_1(-1) + {}^k c_2(-1)^2 + \cdots + {}^k c_k(-1)^k$

$= [1 + (-1)]^k$

$= 0$.

This completes the proof.

Note 1. $\sum_{d|n} \mu(d)$ can be expressed as $\sum_{d|n} \mu(d) = [\frac{1}{n}]$ for $n \geq 1$.

This follows from the relation $[\frac{1}{n}] = 1$ for $n = 1$

$= 0$ for $n > 1$.

Note 2. To illustrate the theorem, let $n = 24$.

The divisors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.

$$\sum_{d|24}\mu(d) = \mu(1)+\mu(2)+\mu(3)+\mu(4)+\mu(6)+\mu(8)+\mu(12)+\mu(24)$$
$$= 1+(-1)+(-1)+0+1+0+0+0 = 0.$$

Theorem 3.7.3. If n be a positive integer, $\phi(n) = \sum_{d|n}\mu(d)\frac{n}{d}$.

[This is a relation connecting $\mu(n)$ and $\phi(n)$.]

Proof. **Case 1.** $n = 1$.

In this case $d = 1$. $\phi(n) = 1$ and $\sum_{d|n}\mu(d)\frac{n}{d} = \mu(1) = 1$.

The theorem holds.

Case 2. $n > 1$. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_i \geq 1$.

$$\frac{\phi(n)}{n} = (1-\tfrac{1}{p_1})(1-\tfrac{1}{p_2})\cdots(1-\tfrac{1}{p_k})$$
$$= 1+\Sigma\frac{(-1)}{p_1}+\Sigma\frac{(-1)^2}{p_1p_2}+\cdots+\frac{(-1)^k}{p_1p_2\cdots p_k}.$$

Thee are k terms in $\Sigma\frac{(-1)}{p_1}$. Each term can be expressed as $\frac{\mu(d)}{d}$, where d is a prime divisor of n.

Thee are ${}^kc_2$ terms in $\Sigma\frac{(-1)^2}{p_1p_2}$. Each term can be expressed as $\frac{\mu(d)}{d}$, for some divisor d of n where d is the product of two distinct prime divisors of n.

The last term $\frac{(-1)^k}{p_1p_2\cdots p_k}$ can be expressed as $\frac{\mu(d)}{d}$, where d is the product of k distinct prime divisors of n.

This is to note that if a divisor d of n is not square-free, then $\frac{\mu(d)}{d} = 0$.

Thus the sum $1+\Sigma\frac{(-1)}{p_1}+\Sigma\frac{(-1)^2}{p_1p_2}+\cdots+\frac{(-1)^k}{p_1p_2\cdots p_k}$ can be expressed as $\sum_{d|n}\frac{\mu(d)}{d}$. Therefore $\phi(n) = \sum_{d|n}\mu(d)\frac{n}{d}$.

This completes the proof.

Note. The theorem gives a summation formula for $\phi(n)$.

To illustrate, (i) let $n = p$, a prime;

$$\phi(n) = \sum_{d|n}\frac{\mu(d)}{d}n$$
$$= n[\tfrac{\mu(1)}{1}+\tfrac{\mu(p)}{p}] = p(1-\tfrac{1}{p});$$

(ii) let $n = pq$, where p, q are primes;

$$\phi(n) = \sum_{d|n}\frac{\mu(d)}{d}n$$
$$= n[\tfrac{\mu(1)}{1}+\tfrac{\mu(p)}{p}+\tfrac{\mu(q)}{q}+\tfrac{\mu(pq)}{pq}] = pq[1+\tfrac{-1}{p}+\tfrac{-1}{q}+\tfrac{1}{pq}]$$
$$= pq(1-\tfrac{1}{p})(1-\tfrac{1}{q});$$

(iii) let $n = p^2q$, where p, q are primes;

$$\phi(n) = \sum_{d|n}\frac{\mu(d)}{d}n$$
$$= p^2q[\tfrac{\mu(1)}{1}+\tfrac{\mu(p)}{p}+\tfrac{\mu(p^2)}{p^2}+\tfrac{\mu(q)}{q}+\tfrac{\mu(pq)}{pq}+\tfrac{\mu(p^2q)}{p^2q}]$$
$$= p^2q[1+\tfrac{-1}{p}+0+\tfrac{-1}{q}+\tfrac{1}{pq}+0]$$
$$= p^2q(1-\tfrac{1}{p})(1-\tfrac{1}{q}).$$

Worked Examples.

1. For each positive integer n, show that $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$.

Any positive integer n is one of the forms $4k, 4k+1, 4k+2, 4k+3$, where k is a positive integer.

If $n = 4k$, then $\mu(n) = \mu(4k) = 0$.

If $n = 4k+1$, then $\mu(n+3) = \mu(4(k+1)) = 0$.

If $n = 4k+2$, then $\mu(n+2) = \mu(4(k+1)) = 0$.

If $n = 4k+3$, then $\mu(n+1) = \mu(4(k+1)) = 0$.

Therefore $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$, whatever positive integer n may be.

2. If $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are primes and $\alpha_i \geq 1$, prove that $\sum_{d|n}\mu(d)\tau(d) = (-1)^k$.

Let $d = 1$. Then $\mu(d)\tau(d) = \mu(1)\tau(1) = 1$.

Let $d = p_1$. Then $\mu(d)\tau(d) = \mu(p_1)\tau(p_1) = (-1).2$.

If $d = p_1, p_2, \ldots, p_k$, then $\sum_d \mu(d)\tau(d) = {}^kc_1(-1).2$.

Let $d = p_1p_2$. Then $\mu(d)\tau(d) = \mu(p_1p_2)\tau(p_1p_2) = (-1)^2.2^2$.

If $d = p_1p_2, p_1p_3, \ldots, p_{k-1}p_k$, then $\sum_d \mu(d)\tau(d) = {}^kc_2(-1)^2.2^2$.

If $d = p_1p_2p_3\cdots p_k$, then $\mu(d)\tau(d) = (-1)^k.2^k = {}^kc_k(-1)^k.2^k$.

If d contains a repeated prime divisor, then $\mu(d) = 0$ and therefore $\mu(d)\tau(d) = 0$.

The only non zero terms in $\sum_{d|n}\mu(d)\tau(d)$ come from those divisors d of n which are either primes or the product of distinct primes (i.e., square-free).

Therefore $\sum_{d|n}\mu(d)\tau(d)$
$= 1+{}^kc_1(-1).2+{}^kc_2(-1)^2.2^2+\cdots+{}^kc_k(-1)^k.2^k$
$= (1-2)^k = (-1)^k$.

3.8. Diophantine equation $x^2 + y^2 = z^2$.

A triplet of positive integers x, y, z satisfying the equation $x^2 + y^2 = z^2$ is called a *Pythagorean triplet*. x, y, z represent the sides of a right triangle, z being the hypotenuse and x, y the legs of the triangle.

If x, y, z be a Pythagorean triplet, then kx, ky, kz is also a Pythagorean triplet for all integers $k > 1$ and they represent the sides of a similar triangle.

A Pythagorean triplet x, y, z is called a *primitive Pythagorean triplet* if $gcd(x, y, z) = 1$. A primitive triplet corresponds to the smallest triangle with integer sides among all similar triangles.

For example, $3, 4, 5$; $6, 8, 10$; $9, 12, 15$; ... are all Pythagorean triplets representing similar right triangles, but $3, 4, 5$ is the primitive triplet corresponding to the class of similar triangles.

Some properties of primitive Pythagorean triplets.

1. If x, y, z be a primitive triplet, then x, y, z are pairwise prime to each other.

Proof. $gcd(x, y, z) = 1$. Let $gcd(x, y) = d > 1$. Then $d|x, d|y$. Let p be a prime divisor of d. Then $p|x, p|y$. This implies $p|x^2, p|y^2$ and therefore $p|z^2$ and this again implies $p|z$. This contradicts that $gcd(x, y, z) = 1$. Therefore $gcd(x, y) = 1$.

Similarly, $gcd(y, z) = 1$, $gcd(x, z) = 1$.

2. If x, y, z be a primitive triplet, then z is odd and x, y are of different parity (i.e., one of them is odd and the other is even.)

Proof. Since $gcd(x, y) = 1$, x, y cannot be both even.

Let x, y be both odd. Then x is either of the forms $4n + 1, 4n + 3$. In any case, x^2 is of the form $4n + 1$. Similarly, y^2 is of the form $4n + 1$. So z^2 is of the form $4n + 2$. Since for any integer, odd or even, a perfect square is either of the form $4n$ or of the form $4n + 1$, z^2 cannot be of the form $4n + 2$. Therefore x, y cannot be both odd.

Hence one of x, y is odd and the other is even and therefore z is odd.

To be definite, we shall take x odd, y even and of course z odd in a primitive Pythagorean triplet x, y, z.

Theorem 3.8.1. The primitive Pythagorean triplets x, y, z are given by $x = m^2 - n^2$, $y = 2mn$, $z = m^2 + n^2$, where m, n are positive integers prime to each other with $m > n$ and $m \not\equiv n (\text{mod } 2)$.

Since we have agreed to take y even, let $y = 2k$ where k is a positive integer. $y^2 = z^2 - x^2$ gives $k^2 = (\frac{z+x}{2})(\frac{z-x}{2}) = uv$, say, where $u = \frac{z+x}{2}$ and $v = \frac{z-x}{2}$. $u > 0, v > 0$; $z = u + v$ and $x = u - v$.

$gcd(x, z) = 1 \Rightarrow gcd(u, v) = 1$. Since $u > 0, v > 0$; $gcd(u, v) = 1$ and uv is a perfect square, it follows that u is a perfect square and v is a perfect square. [This follows from canonical representations of u and v.]

Let $u = m^2, v = n^2$ where m, n are positive integers. $gcd(u, v) = 1 \Rightarrow gcd(m, n) = 1$. If m, n are both odd or both even, then u, v are both even and this contradicts $gcd(u, v) = 1$. Therefore one of m, n is odd and the other is even, i.e., $m \not\equiv n (\text{mod } 2)$.

Conversely, let the positive integers m, n with $m > n$ be prime to each other and $m \not\equiv n (\text{mod } 2)$.

Then $x = m^2 - n^2$, $y = 2mn$, $z = m^2 + n^2$ form a Pythagorean triplet, since $x^2 + y^2 = z^2$ holds. We show that $gcd(x, y, z) = 1$.

Let $gcd(x, y, z) = d > 1$. Let p be a prime divisor of d. Then $p|x, p|y, p|z$. $p|x, p|z \Rightarrow p|m^2, p|n^2$. This again implies $p|m, p|n$, since p is a prime. This contradicts $gcd(m, n) = 1$. So $gcd(x, y, z) = 1$ and x, y, z form a primitive Pythagorean triplet.

This completes the proof.

Let us choose positive integers $m = 2, 3, 4, 5 \ldots$ and correspondingly positive integers $n < m$ such that $gcd(m, n) = 1$ and m, n are of opposite parity.

We obtain the following table of primitive Pythagorean triplets corresponding to the smaller values of m, n.

		x	y	z
m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$
2	1	3	4	5
3	2	5	12	13
4	1	15	8	17
4	3	7	24	25
5	2	21	20	29
5	4	9	40	41

We shall obtain different tables of primitive Pythagorean triplets by imposing different conditions on x, y, z.

1. Let y and z be consecutive positive integers. Then $2mn + 1 = m^2 + n^2$. This gives $m - n = 1$, since $m > n$. $m = n + 1 \Rightarrow x = (n + 1)^2 - n^2 = 2n + 1, y = 2n(n + 1), z = 2n^2 + 2n + 1$.

As x is an odd positive integer, We obtain primitive Pythagorean triplets with every odd positive integer as one leg.

We have the following table corresponding to the smaller values of n.

n	x $2n+1$	y $2n(n+1)$	z $2n^2+2n+1$
1	3	4	5
2	5	12	13
3	7	24	25
4	9	40	41
5	11	60	61

Note. In this case, as $z-y=1$, $x^2=(z+y)(z-y)=z+y$.

2. Let x and z be consecutive odd positive integers. Then $m^2+n^2=m^2-n^2+2$. This gives $n=1$. Since m,n are of opposite parity, m must be even. Let $m=2k$, k being an integer. Therefore $x=4k^2-1, y=4k, z=4k^2+1$.

As y is a multiple of 4, We obtain primitive Pythagorean triplets with every multiple of 4 as one leg.

We have the following table corresponding to the smaller values of k.

k	x $4k^2-1$	y $4k$	z $4k^2+1$
1	3	4	5
2	15	8	17
3	35	12	37
4	63	16	65
5	99	20	101

Theorem 3.8.2. The in-radius of a Pythagorean triangle is a positive integer.

Proof. Let x,y be the sides and z be the hypotenuse of a right triangle. The area of the triangle is $\frac{1}{2}xy$ and the semi-perimeter of the triangle is $\frac{1}{2}(x+y+z)$.

The in-radius r of the triangle is $\frac{xy}{x+y+z}$.

Since $x^2+y^2=z^2$, x,y,z can be taken as $x=m^2-n^2, y=2mn, z=m^2+n^2$, where m,n are positive integers of opposite parity with $m>n$ and $gcd(m,n)=1$.

Threfore $r=\frac{mn(m^2-n^2)}{m(m+n)}=n(m-n)$, a positive integer.

Note. $m-n$ is odd and n is prime to $m-n$.

For every positive integer r, there is a Pythagorean triangle with in-radius r. Because we may take $m-n=1$ and $n=r$, i.e., $m=r+1, n=r$. The triangle is given by $x=2r+1, y=2r(r+1), z=2r^2+2r+1$.

There may be other triangles also. If there be an odd prime divisor p of r, then we may take $m-n=p$ and $n=\frac{r}{p}$ giving a different triangle.

Worked Examples.

1. If x,y,z be a primitive Pythagorean triplet, prove that $12|xy$.

$x=m^2-n^2$, $y=2mn$, $z=m^2+n^2$, where m,n are positive integers prime to each other with $m>n$ and m,n are of different parity.

Since one of m,n is even, y is a multiple of 4 and therefore $4|xy$.

If one of m,n is a multiple of 3, y is a multiple of 3 and therefore $3|xy$.

If none of m,n is a multiple of 3 then by Fermat's theorem, $m^2\equiv 1 \pmod 3$ and $n^2\equiv 1 \pmod 3$. Therefore $m^2-n^2\equiv 0 \pmod 3$, i.e., $3|x$ and therefore $3|xy$.

$4|xy$ and $3|xy \Rightarrow 12|xy$, since $gcd(3,4)=1$.

2. If x,y,z be a primitive Pythagorean triplet, prove that $60|xyz$.

$x=m^2-n^2$, $y=2mn$, $z=m^2+n^2$, where m,n are positive integers prime to each other with $m>n$ and m,n are of different parity.

If one of m,n is a multiple of 5, y is a multiple of 5 and therefore $5|xyz$.

If none of m,n is a multiple of 5 then by Fermat's theorem, $m^2\equiv 1 \pmod 5$ and $n^2\equiv 1 \pmod 5$. Therefore $m^2-n^2\equiv 0 \pmod 5$, i.e., $5|x$ and therefore $5|xyz$.

By Ex.1, $12|xyz$. $12|xyz$ and $5|xyz \Rightarrow 60|xyz$, since $gcd(12,5)=1$.

3. If r be an odd prime, show that there are two Pythagorean triangles with in-radius r.

The in-radius r of a Pythagorean triangle whose sides are $x=m^2-n^2, y=2mn, z=m^2+n^2$, where m,n are positive integers of opposite parity with $m>n$ and $gcd(m,n)=1$ is given by $r=n(m-n)$, where $m-n$ is odd and n is prime to $m-n$.

There are two choices. We may take (i) $m-n=1$ and $n=r$ giving a triangle, (ii) $m-n=r$ and $n=1$ giving a different triangle.

Thus there are two Pythagorean triangles with in-radius r.

Exercises 3D

1. (i) Find the highest power of 3 dividing 153!

(ii) Find the highest power of 5 dividing 153!

(iii) Find the highest power of 11 dividing 1000!

2. (i) Find the number of zeros at the right end of the integer 222!.

(ii) Find the number of zeros at the right end of the integer 333!.

3. Prove that the number of zeros at the right end of the integer $(5^{25}-1)!$ is $\frac{5^{25}-101}{4}$.

4. Prove that the highest power of n contained in $(n^r-1)!$ is $\frac{n^r-nr+r-1}{n-1}$.

5. Prove that for every positive real number a, $[\frac{a}{2}]+[\frac{a+1}{2}]=[a]$.

[Hint. Assume (i) $a=2m+f, 0\le f<1$ and (ii) $a=2m+1+f, 0\le f<1$, where m is an integer.**]**

6. If n be a positive integer, prove that

(i) $[\frac{n}{2}]-[\frac{-n}{2}]=n$, (ii) $[x+n]=[x]+n$ for all real x.

7. Prove that for every real number a, $[a]+[a+\frac{1}{3}]+[a+\frac{2}{3}]=[3a]$.

[Hint. Let $a=n+f, 0\le f<1$. Consider the cases (i) $0\le 3f<1$, (ii) $1\le 3f<2$, (iii) $2\le 3f<3$.**]**

8. If $n=p_1p_2\dots p_k$, where $p_1,p_2,\dots,p_k$ are distinct primes, prove that $\sum_{d|n}|\mu(d)|=2^k$.

9. If $n=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, where $p_1,p_2,\dots,p_k$ are distinct primes and $\alpha_i\ge 1$, prove that $\sum_{d|n}\mu(d)\sigma(d)=(-1)^k p_1p_2\dots p_k$.

10. If $n=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, where $p_1,p_2,\dots,p_k$ are distinct primes and $\alpha_i\ge 1$, prove that $\sum_{d|n}\mu(d)\phi(d)=(2-p_1)(2-p_2)\dots(2-p_k)$.

11. Prove that there is only one Pythagorean triangle whose sides are in arithmetic progression.

12. Prove that there is no Pythagorean triangle whose sides are in geometric progression.

13. If there is no odd prime divisor of r, prove that there is only one Pythagorean triangle with in-radius r.

14. If $r=pq$, the product of two distinct primes, show that there are four Pythagorean triangles with in-radius r.

4. POLYNOMIALS

4.1. Polynomials.

An expression of the form

$$a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n,$$

where $a_0,a_1,\dots,a_n$ are given numbers (real or complex), n is a non-negative integer and x is a variable, is called a *polynomial* in x or a *rational integral function* of x.

$a_0,a_1,\dots,a_n$ are called *coefficients* and $a_0x^n, a_1x^{n-1},\dots,a_n$ are called *terms* of the polynomial. If $a_0\neq 0$, the polynomial is said to be of *degree* n and the term a_0x^n is called the *leading term.*

The general form of a polynomial of degree 1 is a_0x+a_1, $a_0\neq 0$; and of degree 2 is $a_0x^2+a_1x+a_2$, $a_0\neq 0$.

A non-zero constant a_0 itself is said to be a *polynomial of degree* 0 while a polynomial all of whose coefficients are zero is said to be a *zero polynomial* and is denoted by 0 and no degree is assigned to it.

Since a polynonial is an expression containing the variable x, it is denoted by $f(x), g(x)$ etc. The value of the polynomial $f(x)$ for $x=a$ where a is a real number or a complex number is denoted by $f(a)$.

In particular, if the coefficients $a_0,a_1,a_2,\dots$ of a polynomial $f(x)$ be all real numbers, the polynomial $f(x)$ is said to be a *real* polynomial.

Equality. Two polynomials of the same degree

$a_0x^n+a_1x^{n-1}+\cdots+a_n$ and $b_0x^n+b_1x^{n-1}+\cdots+b_n$

are said to be *equal or identical* if $a_0=b_0, a_1=b_1,\dots,a_n=b_n$.

Addition. Let $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n, g(x)=b_0x^m+b_1x^{m-1}+\cdots+b_m$ be two polynomials of degree n and m respectively.

The *sum* $f(x)+g(x)$ is a polynomial given by

$$\begin{aligned} f(x)+g(x) &= a_0x^n+\cdots+(a_{n-m}+b_0)x^m+\cdots+(a_n+b_m) \\ &\qquad \text{if } m<n, \\ &= (a_0+b_0)x^n+\cdots+(a_n+b_n) \text{ if } m=n, \\ &= b_0x^m+\cdots+(b_{m-n}+a_0)x^n+\cdots+(b_m+a_n) \\ &\qquad \text{if } m>n. \end{aligned}$$

Multiplication. Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m$ be two polynomials of degree n and m respectively.

The *product* $f(x)g(x)$ is a polynomial of degree $m+n$ given by $f(x)g(x) = c_0x^{m+n} + c_1x^{m+n-1} + \cdots + c_{m+n}$, where $c_i = a_0b_i + a_1b_{i-1} + \cdots + a_ib_0$, taking $a_{n+1} = a_{n+2} = \cdots = a_{n+m} = 0$,

$$b_{m+1} = b_{m+2} = \cdots = b_{m+n} = 0.$$

$c_0 = a_0b_0 \neq 0$. Therefore the degree of $f(x)g(x)$ is $m+n$.

Theorem 4.1.1. Division algorithm.

Let $f(x)$ and $g(x)$ be two polynomials of degree n and m respectively and $n \geq m$. Then there exist two uniquely determined polynomials $q(x)$ and $r(x)$ satisfying

$$f(x) = g(x)q(x) + r(x),$$

where the degree of $q(x)$ is $n-m$ and $r(x)$ is either a zero polynomial or the degree of $r(x)$ is less than m.

Proof. Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \quad a_0 \neq 0$

$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m, \quad b_0 \neq 0.$$

Let us consider the polynomial

$f_1(x) = f(x) - \frac{a_0}{b_0}x^{n-m}g(x) = f(x) - c_0x^{n-m}g(x)$, where $c_0 = \frac{a_0}{b_0}$.

Either $f_1(x)$ is a zero polynomial, or else the degree of $f_1(x)$ is $n_1 < n$.

Now $f(x) = c_0x^{n-m}g(x) + f_1(x)$. Two cases arise.

Case I. $f_1(x)$ is either a zero polynomial, or the degree of $f_1(x) < m$. In this case

$f(x) = q(x)g(x) + r(x)$, where $q(x) = c_0x^{n-m}, r(x) = f_1(x)$ and the theorem is proved.

Case II. $n_1 \geq m$.

In this case $f_1(x) = p_0x^{n_1} + \cdots + p_n. \quad p_0 \neq 0.$

$$\begin{aligned} \text{Let } f_2(x) &= f_1(x) - \frac{p_0}{b_0}x^{n_1-m}g(x) \\ &= f_1(x) - c_1x^{n_1-m}g(x), \text{ where } c_1 = \frac{p_0}{b_0}. \end{aligned}$$

Either $f_2(x)$ is a zero polynomial, or else the degree of $f_2(x)$ is $n_2 < n_1$.

Now $f_1(x) = c_1x^{n_1-m}g(x) + f_2(x)$. Two cases arise.

Case I. $f_2(x)$ is either a zero polynomial, or the degree of $f_2(x)$ is less than m. In this case

$f(x)$ can be expressed as $[c_0x^{n-m} + c_1x^{n_1-m}]g(x) + f_2(x)$

$= q(x)g(x) + r(x)$, proving the theorem.

Case II. $n_2 \geq m$.

Repeating the process we get polynomials $f_1(x), f_2(x), f_3(x), \ldots$ successively whose degrees form a monotone decreasing sequence of integers so that after some steps we obtain a polynomial $f_{k+1}(x)$ which is either a zero polynomial or whose degree n_{k+1} is less than m.

$$\begin{aligned} \text{We have } f(x) &= c_0x^{n-m}g(x) + f_1(x), \\ f_1(x) &= c_1x^{n_1-m}g(x) + f_2(x), \\ &\ldots \quad \ldots \quad \ldots \\ f_k(x) &= c_kx^{n_k-m}g(x) + f_{k+1}(x). \end{aligned}$$

$$\begin{aligned} \text{Therefore } f(x) &= (c_0x^{n-m} + c_1x^{n_1-m} + \cdots + c_kx^{n_k-m})g(x) + f_{k+1}(x) \\ &= q(x)g(x) + r(x), \text{ where } q(x) = c_0x^{n-m} + c_1x^{n_1-m} + \cdots + c_kx^{n_k-m} \text{ and } r(x) = f_{n+1}(x). \end{aligned}$$

We now prove the uniqueness of $q(x)$ and $r(x)$.

If possible, let there be two other polynomials $q_1(x)$ and $r_1(x)$ such that $f(x) = q_1(x)g(x) + r_1(x)$, where the degree of $q_1(x)$ is $n-m$ and $r_1(x)$ is either a zero polynomial, or else the degree of $r_1(x)$ is less than m.

$$\begin{aligned} \text{Then } 0 &= \{q(x) - q_1(x)\}g(x) + \{r(x) - r_1(x)\} \\ \text{or, } r_1(x) - r(x) &= \{q(x) - q_1(x)\}g(x) \quad \ldots \quad \text{(A)} \end{aligned}$$

By the property of $r(x)$ and $r_1(x)$, the left hand side polynomial $r_1(x) - r(x)$ is either a zero polynomial or else the degree of $r_1(x) - r(x)$ is less than that of $g(x)$. But the second possibility is ruled out because the equality (A) demands that the degree of $r_1(x) - r(x)$ is greater than that of $g(x)$.

Therefore $r_1(x) - r(x)$ and consequently $q(x) - q_1(x)$ are zero polynomials and this proves uniqueness of $q(x)$ and $r(x)$.

$g(x)$ is said to be the *divisor*, $q(x)$ is said to be the *quotient* and $r(x)$ is said to be the *remainder*.

If $r(x)$ be a zero polynomial, $f(x)$ is said to be *divisible* by $g(x)$. In this case $g(x)$ is said to be a *factor* of $f(x)$.

If, in particular, $g(x)$ be a polynomial of degree 1, then $r(x)$ is either a zero polynomial or else it is a polynomial of degree 0. That is, in this case the remainder is a constant, zero or non-zero.

Theorem 4.1.2. Remainder theorem.

If a polynomial $f(x)$ is divided by $x - \alpha$ the remainder is $f(\alpha)$.

Proof. Let $q(x)$ be the quotient and $r(x)$ be the remainder when $f(x)$ is divided by $x - \alpha$. Since the divisor $x - \alpha$ is a polynomial of degree 1, the remainder $r(x)$ is a constant, say R.

Therefore $f(x) = (x-\alpha)q(x) + R$.
Hence $f(\alpha) = 0.q(\alpha) + R$. That is, $f(\alpha) = R$.

Corollary. $f(x)$ can be expressed as $f(x) = (x-\alpha)q(x) + f(\alpha)$.

Definition. If $f(x)$ is a polynomial and $f(\alpha) = 0$ then α is said to be a *zero* of the polynomial $f(x)$.

Theorem 4.1.3. Factor theorem.

If $f(x)$ be a polynomial then $x-\alpha$ is a factor of $f(x)$ if and only if $f(\alpha) = 0$.

Proof. Let $x-\alpha$ be a factor of $f(x)$. Then $f(x) = (x-\alpha).g(x)$ for some polynomial $g(x)$. Therefore $f(\alpha) = (\alpha-\alpha).g(\alpha) = 0$.

Conversely, let $f(\alpha) = 0$. Let R be the remainder and $q(x)$ be the quotient when $f(x)$ is divided by $x-\alpha$.

Then $f(x) = (x-\alpha)q(x) + R$. But $R = f(\alpha) = 0$.
Therefore $f(x) = (x-\alpha)q(x)$, showing that $x-\alpha$ is a factor of $f(x)$.

Worked Examples.

1. Prove that x^2+x+1 is a factor of $x^{10}+x^5+1$.

We have $x^2+x+1 = (x-\omega)(x-\omega^2)$, where ω is an imaginary cube root of 1.

Let $f(x) = x^{10}+x^5+1$. Then $f(\omega) = \omega^{10}+\omega^5+1 = \omega+\omega^2+1 = 0$.
$f(\omega^2) = \omega^{20}+\omega^{10}+1 = \omega^2+\omega+1 = 0$.

Therefore $(x-\omega)(x-\omega^2)$, i.e., x^2+x+1 is a factor of $f(x)$.

2. Find the remainder when $4x^5+3x^3+6x^2+5$ is divided by $2x+1$.

Let $f(x) = 4x^5+3x^3+6x^2+5$. Let $q(x)$ be the quotient and R be the remainder when $f(x)$ is divided by $2x+1$.

Then $f(x) = (2x+1)q(x) + R$ and therefore $f(-\frac{1}{2}) = 0.q(-\frac{1}{2}) + R$.
The remainder $R = f(-\frac{1}{2}) = 4.\frac{-1}{32} + 3.\frac{-1}{8} + 6.\frac{1}{4} + 5 = 6$.

3. Find the remainder when $x^5-3x^4+4x^2+x+4$ is divided by $(x+1)(x-2)$.

Let $f(x) = x^5-3x^4+4x^2+x+4$. Let $q(x)$ be the quotient and $rx+s$ be the remainder when $f(x)$ is divided by $(x+1)(x-2)$.

Then $f(x) = (x+1)(x-2)q(x) + (rx+s)$.

Hence $f(-1) = 0.q(-1) + (-r+s)$ and $f(2) = 0.q(2) + (2r+s)$.
But $f(-1) = 3$ and $f(2) = 6$.

Consequently, $-r+s = 3$ and $2r+s = 6$. Therefore $r = 1, s = 4$ and the remainder is $x+4$.

4. For what integral values of m, x^2+x+1 is a factor $x^{2m}+x^m+1$?

$x^2+x+1 = (x-\omega)(x-\omega^2)$, where ω is an imaginary cube root of 1.
Let $f(x) = x^{2m}+x^m+1$. Then
$f(\omega) = \omega^{2m}+\omega^m+1$, $f(\omega^2) = \omega^{4m}+\omega^{2m}+1$.
x^2+x+1 is a factor of $x^{2m}+x^m+1$ if $f(\omega) = 0$ and $f(\omega^2) = 0$.
The integer m is one of the forms $3k, 3k+1, 3k+2$ where k is an integer.

When $m = 3k$, $f(\omega) = 1+1+1 = 3$ and $f(\omega^2) = 1+1+1 = 3$.
When $m = 3k+1$, $f(\omega) = \omega^2+\omega+1 = 0$ and $f(\omega) = \omega^2+\omega+1 = 0$.
When $m = 3k+2$, $f(\omega) = \omega+\omega^2+1 = 0$ and $f(\omega^2) = \omega^2+\omega+1 = 0$.

Therefore if m be not an integral multiple of 3, x^2+x+1 is a factor of $x^{2m}+x^m+1$.

4.2. Synthetic division.

We have seen that when a polynomial $f(x)$ is divided by $x-\alpha$ the remainder can be obtained readily without going into laborious process of division algorithm.

We now develop a simple method of obtaining the quotient when $f(x)$ is divided by $x-\alpha$.

Let the quotient be $b_0x^{n-1}+b_1x^{n-2}+\cdots+b_{n-1}$ and the remainder be R.

Then $a_0x^n+a_1x^{n-1}+\cdots+a_n = (b_0x^{n-1}+b_1x^{n-2}+\cdots+b_{n-1})(x-\alpha)+R$. Equating coefficients of like powers of x, we have

$a_0 = b_0$
$a_1 = b_1 - \alpha b_0$
$a_2 = b_2 - \alpha b_1$
... ...
$a_{n-1} = b_{n-1} - \alpha b_{n-2}$
$a_n = R - \alpha b_{n-1}$.

Therefore $b_0, b_1, \ldots, b_{n-1}$ and R are given by
$b_0 = a_0, b_1 = a_1 + \alpha b_0, b_2 = a_2 + \alpha b_1, \ldots, b_{n-1} = a_{n-1} + \alpha b_{n-2}, R = a_n + \alpha b_{n-1}$.

The calculation for $b_0, b_1, \ldots, b_{n-1}$ and R can be performed in the following scheme.

a_0	a_1	a_2	...	a_{n-1}	a_n
	αb_0	αb_1	...	αb_{n-2}	αb_{n-1}
b_0	b_1	b_2	...	b_{n-1}	R

The coefficients of the polynomial $f(x)$ are written in the first line; if any term be absent the zero coefficient should be included in the sequence. The third line begins with $b_0 (= a_0)$.

b_0 is multiplied by α and is written below a_1, the sum $a_1 + \alpha b_0$ gives b_1. b_1 is multiplied by α and written below a_2, the sum $a_2 + \alpha b_1$ gives b_2,...

The process continues till at the last step b_{n-1} is multiplied by α and is written below a_n, the sum $a_n + \alpha b_{n-1}$ gives the remainder R.

The method described in the scheme is called the *method of synthetic division*. The method gives a ready calculation for the remainder $f(\alpha)$.

We shall see later that the method also determines quickly the values of the derivatives of successive orders of $f(x)$ at $x = \alpha$, i.e., the values of $f'(\alpha), f''(\alpha), \ldots, f^n(\alpha)$.

Application of the method.

(i) To express a polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ as a polynomial in $x - \alpha$.

Let $f(x)$ be expressed as $A_0(x-\alpha)^n + A_1(x-\alpha)^{n-1} + \cdots + A_n$.

Therefore $f(x) = (x-\alpha)[A_0(x-\alpha)^{n-1} + A_1(x-\alpha)^{n-2} + \cdots + A_{n-1}] + A_n$

This shows that if $f(x)$ is divided by $x - \alpha$, the quotient $q_1(x)$ is $A_0(x-\alpha)^{n-1} + A_1(x-\alpha)^{n-2} + \cdots + A_{n-1}$ and the remainder is A_n.

Again $q_1(x) = (x-\alpha)[A_0(x-\alpha)^{n-2} + A_1(x-\alpha)^{n-3} + \cdots + A_{n-2}] + A_{n-1}$

This shows that if $q_1(x)$ is divided by $x - \alpha$, the quotient $q_2(x)$ is $A_0(x-\alpha)^{n-2} + A_1(x-\alpha)^{n-3} + \cdots + A_{n-2}$ and the remainder is A_{n-1}.

The process can be continued n times and the remainders are successively $A_n, A_{n-1}, \ldots, A_1$. Finally $a_0 = A_0$.

(ii) Let $f(x)$ be a polynomial in x. To express $f(x + \alpha)$ as a polynomial in x.

Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$.
Let us first express $f(x)$ as $f(x) = A_0(x-\alpha)^n + A_1(x-\alpha)^{n-1} + \cdots + A_n$.
Then $f(x+\alpha) = A_0x^n + A_1x_{n-1} + \cdots + A_n$.

Worked Examples.

1. Expand $f(x) = x^4 - 4x^3 + 3x^2 + 3x + 7$ as a polynomial in $x - 1$.

1	1	-4	3	3	7
		1	-3	0	3
	1	-3	0	3	10
		1	-2	-2	
	1	-2	-2	1	
		1	-1		
	1	-1	-3		
		1			
	1	0			

Hence $f(x) = 1(x-1)^4 + 0(x-1)^3 - 3(x-1)^2 + 1(x-1) + 10$.

2. $f(x) = x^4 - x^3 + 2x^2 + 6x - 2$. Find $f(x+2)$.

Let $f(x) = A_0(x-2)^4 + A_1(x-2)^3 + \cdots + A_4$. Then $A_4, A_3, \ldots, A_0$ can be calculated by sussessive application of the method of synthetic division as in the following scheme.

2	1	-1	2	6	-2
		2	2	8	28
	1	1	4	14	26
		2	6	20	
	1	3	10	34	
		2	10		
	1	5	20		
		2			
	1	7			

Hence $f(x) = 1(x-2)^4 + 7(x-2)^3 + 20(x-2)^2 + 34(x-2) + 26$ and consequently, $f(x+2) = x^4 + 7x^3 + 20x^2 + 34x + 26$.

Theorem 4.2.1. Taylor's theorem.

If $f(x)$ is a polynomial in x of degree n and α is any number, real or complex,

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \cdots + \frac{f^n(\alpha)}{n!}(x-\alpha)^n.$$

Proof. Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ where $a_0 \neq 0$ and let us assume that $f(x)$ is expressed as $A_n + A_{n-1}(x-\alpha) + A_{n-2}(x-\alpha)^2 + \cdots + A_0(x-\alpha)^n$. Then $f(\alpha) = A_n$.

$f'(x) = A_{n-1} + 2A_{n-2}(x-\alpha) + \cdots + nA_0(x-\alpha)^{n-1}$.
Therefore $f'(\alpha) = A_{n-1}$.

$f''(x) = 2.1A_{n-2} + 3.2A_{n-3}(x-\alpha) + \cdots + n(n-1)A_0(x-\alpha)^{n-2}$
Therefore $f''(\alpha) = 2!A_{n-2}$, i.e., $A_{n-2} = \frac{f''(\alpha)}{2!}$.

... ...

$f^n(x) = n!A_0$. Therefore $f^n(\alpha) = n!A_0$, i.e., $A_0 = \frac{f^n(\alpha)}{n!}$.

Hence $f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \cdots + \frac{f^n(\alpha)}{n!}(x-\alpha)^n$.

Note. The coefficients $A_0, A_1, \ldots, A_n$ are obtained as remainders in the successive application of the method of synthetic division shown in the scheme of Example 1. The successive derivatives at $x = \alpha$ i.e., $f'(\alpha), f''(\alpha), \ldots, f^n(\alpha)$ are therefore readily obtained from those remainders.

4.3. Zero of a polynomial.

Definition. α is said to be a *zero of order* r of the polynomial $f(x)$ if $(x-\alpha)^r$ is a factor of $f(x)$ while $(x-\alpha)^{r+1}$ is not a factor of $f(x)$.

Theorem 4.3.1. α is a zero of order r of the polynomial $f(x)$ if and only if $f(\alpha) = f'(\alpha) = \cdots = f^{r-1}(\alpha) = 0$, and $f^r(\alpha) \neq 0$.

Proof. Let α be a zero order r. Then $(x-\alpha)^r$ is a factor of $f(x)$ but $(x-\alpha)^{r+1}$ is not. Let $f(x) = (x-\alpha)^r\phi(x)$ where $\phi(\alpha) \neq 0$. Because $\phi(\alpha) = 0$ implies that $x-\alpha$ is a factor of $\phi(x)$ and consequently $(x-\alpha)^{r+1}$ is a factor of $f(x)$, a contradiction.

$$\begin{aligned} f'(x) &= (x-\alpha)^r\phi'(x) + r(x-\alpha)^{r-1}\phi(x) \\ &= (x-\alpha)^{r-1}[(x-\alpha)\phi'(x) + r\phi(x)] \\ &= (x-\alpha)^{r-1}\psi(x), \text{ where } \psi(x) = (x-\alpha)\phi'(x) + r\phi(x) \\ &\qquad \text{and } \psi(\alpha) = r\phi(\alpha) \neq 0. \end{aligned}$$

Therefore $f'(\alpha) = 0$.

$f''(x) = (x-\alpha)^{r-2}[(x-\alpha)\psi'(x) + (r-1)\psi(x)]$

$= (x-\alpha)^{r-2}\gamma(x)$, where $\gamma(x) = (x-\alpha)\psi'(x) + (r-1)\psi(x)$ and $\gamma(\alpha) \neq 0$.

Therefore $f''(\alpha) = 0$.

... ...

$f^{r-1}(x) = (x-\alpha)g(x)$ where $g(\alpha) \neq 0$. Therefore $f^{r-1}(\alpha) = 0$.

$f^r(x) = (x-\alpha)g'(x) + g(x)$. Therefore $f^r(\alpha) \neq 0$.

Conversely, let $f(\alpha) = f'(\alpha) = \cdots = f^{r-1}(\alpha) = 0, f^r(\alpha) \neq 0$.

By Taylor's theorem,

$$\begin{aligned} f(x) &= f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \cdots + \frac{f^n(\alpha)}{n!}(x-\alpha)^n \\ &= \frac{f^r(\alpha)}{r!}(x-\alpha)^r + \cdots + \frac{f^n(\alpha)}{n!}(x-\alpha)^n \\ &= (x-\alpha)^r\left[\frac{f^r(\alpha)}{r!} + \frac{f^{r+1}(\alpha)}{(r+1)!}(x-\alpha) + \cdots + \frac{f^n(\alpha)}{n!}(x-\alpha)^{n-r}\right] \\ &= (x-\alpha)^r\phi(x) \text{ where } \phi(\alpha) = \frac{f^r(\alpha)}{r!} \neq 0. \end{aligned}$$

This proves that α is a zero of order r.

Worked Examples.

1. $x^3 + 3px + q$ has a factor of the form $(x-\alpha)^2$. Show that $q^2 + 4p^3 = 0$.

Let $f(x) = x^3 + 3px + q$.
Since $(x-\alpha)^2$ is a factor of $f(x)$, $f(\alpha) = 0, f'(\alpha) = 0$.

That is, $\alpha^3 + 3p\alpha + q$ and $3\alpha^2 + 3p = 0$.

Therefore $p = -\alpha^2, q = 2\alpha^3$.
Eliminating α, we have $q^2 + 4p^3 = 0$.

2. If $x^4 + px^2 + qx + r$ has a factor of the form $(x-\alpha)^3$, show that $8p^3 + 27q^2 = 0$ and $p^2 + 12r = 0$.

Let $f(x) = x^4 + px^2 + qx + r = 0$.

Since $(x-\alpha)^3$ is a factor of $f(x)$, $f(\alpha) = f'(\alpha) = f''(\alpha) = 0$.

Therefore $\alpha^4 + p\alpha^2 + q\alpha + r = 0$, $4\alpha^3 + 2p\alpha + q = 0$, $12\alpha^2 + 2p = 0$.

Therefore $p = -6\alpha^2, q = 8\alpha^3, r = -3\alpha^4$.

Eliminating α, we have $(\frac{p}{-6})^3 = (\frac{q}{8})^2$ and $(\frac{p}{-6})^2 = \frac{r}{-3}$.
Hence $8p^3 + 27q^2 = 0$ and $p^2 + 12r = 0$.

Exercises 4

1. Find the remainder when

(i) $x^5 + 5x^4 + x^3 + 5x^2 + 2x + 11$ is divided by $x + 5$,

(ii) $2x^5 + x^4 + 2x^3 + x^2 + 4x + 3$ is divided by $2x + 1$.

2. Find the quotient and remainder when

(i) $x^6 + x^3 + 1$ is divided by $x + 1$,

(ii) $2x^4 + 7x^3 + x^2 + x + 4$ is divided by $2x + 1$.

3. Show that

(i) $x^{20} + x^{15} + x^{10} + x^5$ is divisible by $x^2 + 1$,

(ii) $x^{20} + x^{10} + 1$ is divisible by $x^2 - x + 1$,

(iii) $(x+1)^{16} + x^{16} + 1$ is divisible by $x^2 + x + 1$,

(iv) $(x+1)^7 - x^7 - 1$ is divisible by $x(x+1)(x^2 + x + 1)$.

4. (i) Express $x^5 - 5x^4 + 12x^2 - 1$ as a polynomial in $(x-1)$.

(ii) If $f(x) = x^4 - 3x^3 + 10x^2$, express $f(x+3)$ as polynomial in x.

(iii) If $f(x) = x^3 + 6x^2 + 12x - 19$, express $f(x+h)$ as polynomial in x. Determine h so that $f(x+h)$ is free from the term containing x^2.

5. Find the remainder when

(i) $x^{10} + x^7 + x^4 + x^3 + 1$ is divided by $x^2 + 1$,

(ii) $x^4 - 3x^3 + 2x^2 + x - 1$ is divided by $x^2 - 4x + 3$,

(iii) $x^6 + 2x^5 - 3x^3 + 2x^2 - 5x + 10$ is divided by $(x-1)^2$,

(iv) $x^{10} + 1$ is divided by $(x^2 + 1)(x^2 + x + 1)$,

(v) $x^{10} + 1$ is divided by $(x+1)(x^2 + x + 1)$.

6. A polynomial $f(x)$ leaves the remainder 10 when it is divided by $(x-2)$ and the remainder $2x - 3$ when it is divided by $(x+1)^2$. Find the remainder when it is divided by $(x-2)(x+1)^2$.

7. $f(x)$ is a polynomial of degree 4 and $f(n) = n + 1$ for $n = 1, 2, 3, 4$. If $f(0) = 25$, find $f(5)$.

8. If $f(x)$ be a polynomial in x and a, b are unequal, show that the remainder in the division of $f(x)$ by $(x-a)(x-b)$ is $\frac{(x-b)f(a)-(x-a)f(b)}{a-b}$.

9. Show that $1 - \frac{x}{1!} + \frac{x(x-1)}{2!} - \frac{x(x-1)(x-2)}{3!} + \cdots + (-1)^n \frac{x(x-1)\ldots(x-n+1)}{n!}$
$= \frac{(-1)^n}{n!}(x-1)(x-2)\ldots(x-n)$.

10. If the polynomial $x^n - qx^{n-m} + r$ has a factor of the form $(x-\alpha)^2$, show that $[\frac{q}{n}(n-m)]^n = [\frac{r}{m}(n-m)]^m$.

11. If $x^2 + px + 1$ be a factor of $ax^3 + bx + c$, prove that $a^2 - c^2 = ab$. Show that in this case $x^2 + px + 1$ is also a factor of $cx^3 + bx^2 + a$.

12. If $ax^4 + bx^3 + c$ has a factor of the form $x^2 + kx + 1$, prove that $(a+c)(a-c)^2 = b^2c$.

13. Prove that $x^2 + px + p^2$ is a factor of $(x+p)^n - x^n - p^n$, if n be odd and not divisible by 3.

14. Prove that $x^2 + y^2 + z^2 - xy - yz - zx$ is a factor of $(x-y)^n + (y-z)^n + (z-x)^n$, if n is not divisible by 3.

5. THEORY OF EQUATIONS

5.1. Algebraic equations.

Let $f(x)$ be a polynomial in x of degree ≥ 1 whose co-efficients are real or complex numbers.

Then $f(x) = 0$ is said to be an *algebraic equation* or a *polynomial equation*. The degree of the polynomial $f(x)$ is said to be the *degree* of the equation $f(x) = 0$.

An equation of degree 1 is $a_0x + a_1 = 0$, where $a_0 \neq 0$.

An equation of degree 2 is $a_0x^2 + a_1x + a_2 = 0$, where $a_0 \neq 0$. This is said to be a *quadratic equation.*

An equation of degree 3 is $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$, where $a_0 \neq 0$. This is said to be a *cubic equation.*

An equation of degree 4 is $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$, where $a_0 \neq 0$. This is said to be a *biquadratic equation.*

The general form of an algebraic equation of degree n is $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, where $a_0, a_1, \ldots, a_n$ are real or complex numbers and $a_0 \neq 0$.

If α be a value of x for which $f(x)$ becomes zero, i.e., if $f(\alpha) = 0$, then α is said to be a *root* of the equation $f(x) = 0$.

The existence of such an α for which $f(x)$ becomes zero is assured by a theorem, called the **Fundamental theorem of classical algebra,** which states that

every algebraic equation has a root, real or complex.

Here we take the theorem without proof and derive some important deductions from it.

Theorem 5.1.1. An algebraic equation of degree n has n roots and no more.

Proof. Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ be a polynomial with co-efficients real or complex, of degree n. Then $a_0 \neq 0$.

The equation $f(x) = 0$ is an algebraic equation of degree n.

By the fundamental theorem, this equation has a root, say α_1.

Then $f(\alpha_1) = 0$ and by the factor theorem, $x - \alpha_1$ is a factor of the polynomial $f(x)$.

Let $f(x) = (x - \alpha_1)f_1(x)$ where $f_1(x)$ is a polynomial of degree $n-1$ with leading co-efficient a_0. By the fundamental theorem, the equation $f_1(x) = 0$ has a root, say α_2.

Then $f(\alpha_2) = 0$ and by the factor theorem, $x - \alpha_2$ is a factor of the polynomial $f_1(x)$.

Let $f_1(x) = (x-\alpha_2)f_2(x)$ where $f_2(x)$ is a polynomial of degree $n-2$ with leading co-efficient a_0.

If $n > 2$, we continue with the same reasoning and come to some polynomial $f_{n-1}(x) = a_0(x - \alpha_n)$.

$$\begin{aligned} \text{Therefore } f(x) &= (x-\alpha_1)f_1(x) \\ &= (x-\alpha_1)(x-\alpha_2)f_2(x) \\ &\quad \dots \quad \dots \\ &= (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{n-1})f_{n-1}(x) \\ &= a_0(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{n-1})(x-\alpha_n). \end{aligned}$$

This shows that $f(x)$ is expressed as the product of n linear factors, each factor corresponds to a root and this proves that $\alpha_1, \alpha_2, \dots, \alpha_n$ are n roots of the equation $f(x) = 0$.

Now we shall prove that there cannot be more than n roots. If possible, let β be a root of the equation $f(x) = 0$ where β is different from $a_1, a_2, \dots, a_n$. Since β is a root, $f(\beta) = 0$ and this would imply $a_0(\beta - \alpha_1)(\beta - \alpha_2)\dots(\beta - \alpha_n) = 0$.

This is impossible because $a_0 \neq 0$ and $\beta - \alpha_i \neq 0$ for $i = 1, 2, \dots, n$. Therefore $f(x) = 0$ cannot have more than n roots.

This completes the proof.

Theorem 5.1.2. If two polynomials $f(x)$ and $g(x)$, both of degree n take equal values for more than n distinct values of x, then $f(x)$ and $g(x)$ are identical polynomials.

Proof. Let $f(x) = g(x)$ for $x = \alpha_1, \alpha_2, \dots, a_m$ where $m > n$.

Then $f(\alpha_1) = g(\alpha_1), f(\alpha_2) = g(\alpha_2), \dots, f(\alpha_m) = g(\alpha_m)$.

Let $\phi(x) = f(x) - g(x)$. Then $\phi(x)$ is either a zero polynomial or a polynomial of degree $\leq n$.

In the case of the second possibility, $\phi(x) = 0$ happens to be an equation having m roots $\alpha_1, \alpha_2, \dots, \alpha_m$, since $\phi(\alpha_1) = 0, \phi(\alpha_2) = 0, \dots, \phi(\alpha_m) = 0$. Again since $m > n$, the degree of the equation $\phi(x) = 0$ cannot be n or less than n, i.e., the degree of the polynomial $\phi(x)$ cannot be $\leq n$, a contradiction.

Therefore $\phi(x)$ is a zero polynomial and this proves $f(x)$ and $g(x)$ are identical polynomials.

This completes the proof.

Definition. If $f(x)$ be a zero polynomial then $f(x) = 0$ is said to be an *identity*. In other words, if $f(x)$ and $g(x)$ be two identical polynomials then $f(x) = g(x)$ is said to be an *identity*.

Examples.

1. $(a-b)[(a+b)x^2 + x + 1] + (b-c)[(b+c)x^2 + x + 1] + (c-a)[(c+a)x^2 + x + 1] = 0$ is an identity, because the left hand side polynomial is a zero polynomial.

2. $a(x^2+bx+1)+b(x^2+cx+1)+c(x^2+ax+1) = a(x^2+cx+1)+b(x^2+ax+1)+c(x^2+bx+1)$ is an identity, because the equality is satisfied by $x = a, x = b, x = c$.

We have seen that if $f(x) = 0$ is a polynomial equation of degree n then the equation has n roots, say $\alpha_1, \alpha_2, \dots, \alpha_n$. In this case $f(x)$ can be expressed as the product of n linear factors $a_0(x-\alpha_1)(x-\alpha_2)\dots(x-a_n)$, a_0 being the leading co-efficient of the polynomial $f(x)$. The factors need not be all distinct. If may be that $a_1 = \alpha_2 = \dots = \alpha_r$. In this case $(x-\alpha_1)^r$ is a factor of $f(x)$ and α_1 is said to be a *multiple root of order* r. There may be many multiple roots of $f(x) = 0$. If however, each root is counted according to its multiplicity, the number of roots of the equation $f(x) = 0$ is exactly n.

If α be a root of the polynomial equation $f(x) = 0$ of multiplicity r, then $(x-\alpha)^r$ is a factor of $f(x)$.

Then $f(x) = (x-\alpha)^r\phi(x)$, where $\phi(x)$ is a polynomial of degree $n-r$ and $\phi(\alpha) \neq 0$, because $\phi(\alpha) = 0$ would imply $x-\alpha$ is a factor of $\phi(x)$ and as such $(x-\alpha)^{r+1}$ would be a factor of $f(x)$ contradicting the multiplicity of α.

Theorem 5.1.3. If α be a multiple root of the polynomial equation $f(x) = 0$ of order r, then α is a multiple root of the equation $f'(x) = 0$ of order $r-1$.

Proof. Let $f(x) = 0$ be an equation of degree n. Since α is a multiple root of order r, $r \leq n$. $f(x)$ can be expressed as $f(x) = (x-\alpha)^r\phi(x)$, where $\phi(x)$ is a polynomial of degree $n-r$ and $\phi(\alpha) \neq 0$.

$$\begin{aligned} f'(x) &= r(x-\alpha)^{r-1}\phi(x) + (x-\alpha)^r\phi'(x) \\ &= (x-\alpha)^{r-1}[r\phi(x) + (x-\alpha)\phi'(x)] \\ &= (x-\alpha)^{r-1}\psi(x), \text{ where } \psi(x) = r\phi(x) + (x-\alpha)\phi'(x) \end{aligned}$$

and $\psi(\alpha) = r\phi(\alpha) \neq 0$.

This proves that α is multiple root of $f'(x) = 0$ of order $r-1$.

Note 1. α is a multiple root of $f''(x) = 0$ of order $r-2$, a multiple root of $f'''(x) = 0$ of order $r-3, \ldots$, a simple root of $f^{r-1}(x) = 0$ and α is not a root of $f^r(x) = 0$.

2. If α is a multiple root of $f(x) = 0$ of order r, then the h.c.f. of the polynomials $f(x)$ and $f'(x)$ contains $(x-\alpha)^{r-1}$ as a factor. This gives a method of determining multiple roots of $f(x) = 0$.

To determine the multiple roots of an equation $f(x) = 0$, we find out the h.c.f. of the polynomials $f(x)$ and $f'(x)$. The zeroes of the h.c.f. polynomial give the multiple roots of $f(x) = 0$.

Worked Examples.

1. The equation $ax^3 + 3bx^2 + 3cx + d = 0$ has two equal roots. Prove that $(bc-ad)^2 = 4(b^2-ac)(c^2-bd)$ and the equal root is $\frac{1}{2}\frac{bc-ad}{ac-b^2}$.

Let $f(x) = ax^3 + 3bx^2 + 3cx + d$ and α be a double root of $f(x) = 0$.
Then $f(\alpha) = 0$ and $f'(\alpha) = 0$.

Therefore $a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0 \quad \ldots$ (i)
and $a\alpha^2 + 2b\alpha + c = 0 \quad \ldots$ (ii)

Multiplying (ii) by α and subtracting from (i), we have
$b\alpha^2 + 2c\alpha + d = 0 \quad \ldots$ (iii)

From (ii) and (iii) $\frac{\alpha^2}{2(bd-c^2)} = \frac{\alpha}{bc-ad} = \frac{1}{2(ac-b^2)}$.

Therefore $\frac{2(bd-c^2)}{bc-ad} = \frac{bc-ad}{2(ac-b^2)}$

or, $(bc-ad)^2 = 4(b^2-ac)(c^2-bd)$. Also $\alpha = \frac{bc-ad}{2(ac-b^2)}$.

2. Determine the multiple roots of the equation $x^5 + 2x^4 + 2x^3 + 4x^2 + x + 2 = 0$.

Let $f(x) = x^5 + 2x^4 + 2x^3 + 4x^2 + x + 2$.
Then $f'(x) = 5x^4 + 8x^3 + 6x^2 + 8x + 1$.

The h.c.f. of $f(x)$ and $f'(x)$ is $x^2 + 1 = (x+i)(x-i)$.

Therefore the equation $f(x) = 0$ has two multiple roots, i of order 2 and $-i$ of order 2.

5.2. Polynomial equations with real coefficients.

We shall discuss some properties of polynomial equations whose coefficients are all real.

Theorem 5.2.1. If an equation *with real coefficients* has a complex root $\alpha + i\beta$ then it has also the conjugate complex root $\alpha - i\beta$.

In other words, in an equation *with real coefficients* imaginary roots occur in conjugate pairs.

Proof. Let $f(x) = 0$ be an equation of degree n with real coefficients and let $\alpha + i\beta$ be a root of $f(x) = 0$. It is obvious that $n \geq 2$.

Let us divide $f(x)$ by the product $\{x - (\alpha + i\beta)\}\{x - (\alpha - i\beta)\}$, i.e., by $(x-\alpha)^2 + \beta^2$. Let $q(x)$ be the quotient and $r(x)$ be the remainder. Then the degree of $q(x)$ is $n-2$ and the degree of $r(x)$ is at most one.

Since $f(x)$ is a real polynomial and $(x-\alpha)^2 + \beta^2$ is also a real quadratic, $q(x)$ and $r(x)$ are both real polynomials and we assume $r(x) = ax + b$ where a and b are both real.

Therefore $f(x) = [(x-\alpha)^2 + \beta^2]q(x) + ax + b$.

Since $\alpha + i\beta$ is a root, $f(\alpha + i\beta) = 0$, i.e., $a(\alpha + i\beta) + b = 0$
or, $(a\alpha + b) + ia\beta = 0$ and this implies $a\alpha + b = 0, a\beta = 0$.

But $\beta \neq 0$. Therefore $a = 0$ and consequently, $b = 0$.
So $f(x) = [(x-\alpha)^2 + \beta^2]q(x)$.

Now $f(\alpha - i\beta) = [(\alpha - i\beta - \alpha)^2 + \beta^2]q(\alpha - i\beta) = 0$, and this proves that $\alpha - i\beta$ is a root of the equation $f(x) = 0$.

This completes the proof.

Note. If $\alpha + i\beta$ be a multiple complex root of $f(x) = 0$ whose coefficients are *all real*, of multiplicity r then $\alpha - i\beta$ is also a multiple root of the equation $f(x) = 0$ of multiplicity r.

From the theorem it follows that imaginary roots of a polynomial equation with real coefficients occur always in pairs. Therefore in such an equation the number of imaginary roots is always even.

If the degree of a polynomial equation with real coefficients be odd then it follows that the equation has at least a real root. If however, the degree be even then the equation may not have a real root at all.

If $f(x)$ be a polynomial with real coefficients, to each linear factor $x - \alpha - i\beta$ of $f(x)$ corresponding to an imaginary root $\alpha + i\beta$ there corresponds another linear factor $x - \alpha + i\beta$ corresponding to the conjugate root $\alpha - i\beta$ and the product $(x-\alpha-i\beta)(x-\alpha+i\beta) = (x-\alpha)^2 + \beta^2$ is a real quadratic factor of $f(x)$.

Therefore we can say that a real polynomial can always be expressed as the product of real linear and real quadratic factors.

Worked Examples.

1. Solve the equation $x^4+x^2-2x+6=0$, it is given that $1+i$ is a root.

Let $f(x)=x^4+x^2-2x+6$.

Since $f(x)=0$ is an equation with real coefficients and $1+i$ is a root of the equation, $1-i$ is also a root.

Therefore $(x-1-i)(x-1+i)=x^2-2x+2$ is a factor of $f(x)$.

Let $f(x)=(x^2-2x+2)q(x)$. Then $q(x)=x^2+2x+3$.

$q(x)=0$ gives $x=-1\pm\sqrt{2}i$.

Therefore the roots of the equation are $1\pm i, -1\pm\sqrt{2}i$.

2. Prove that the roots of the equation $\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}=x$ are all real.

Let us assume that the equation has a root $\alpha+i\beta$ where α,β are real. Since the coefficients of the equation are all real, $\alpha-i\beta$ is another root of the equation.

Since $\alpha+i\beta$ is a root, $\frac{1}{\alpha+i\beta-1}+\frac{2}{\alpha+i\beta-2}+\frac{3}{\alpha+i\beta-3}=\alpha+i\beta$.

Since $\alpha-i\beta$ is a root, $\frac{1}{\alpha-i\beta-1}+\frac{2}{\alpha-i\beta-2}+\frac{3}{\alpha-i\beta-3}=\alpha-i\beta$.

On subtraction, we have

$\left(\frac{1}{\alpha-i\beta-1}-\frac{1}{\alpha-i\beta-1}\right)+2\left(\frac{1}{\alpha+i\beta-2}-\frac{1}{\alpha-i\beta-2}\right)+3\left(\frac{1}{\alpha+i\beta-3}-\frac{1}{\alpha-i\beta-3}\right)=2i\beta$

or, $-2i\beta\left[\frac{1}{(\alpha-1)^2+\beta^2}+\frac{2}{(\alpha-2)^2+\beta^2}+\frac{3}{(\alpha-3)^2+\beta^2}+1\right]=0$.

Since the expression within the bracket is positive, $\beta=0$. This proves that the equation cannot have a root $\alpha+i\beta$ where $\beta\neq 0$. In other words, the roots of the equation are all real.

3. Solve the equation $x^4-x^3+2x^2-x+1=0$ which has four distinct roots of equal moduli.

Let r be the modulus. Two possibilities may occur.

I. Two roots real and two complex. The roots are $r,-r,r(\cos\theta\pm i\sin\theta)$.

II. All complex roots. The roots are $r(\cos\theta\pm i\sin\theta)$ and $r(\cos\phi\pm i\sin\phi)$.

Case I. The given equation is identical with

$(x-r)(x+r)(x^2-2r\cos\theta x+r^2)=0$

or, $x^4-2r\cos\theta x^3+2r^3\cos\theta x-r^4=0$.

This cannot happen, since there is a term containing x^2 in the given equation.

Case II. The given equation is identical with

$(x^2-2r\cos\theta x+r^2)(x^2-2r\cos\phi x+r^2)=0$

or, $x^4-2r(\cos\theta+\cos\phi)x^3+2r^2(1+2\cos\theta\cos\phi)x^2-2r^3(\cos\theta+\cos\phi)x+r^4=0$.

Therefore $2r(\cos\theta+\cos\phi)=1$, $2r^2(1+2\cos\theta\cos\phi)=2$, $2r^3(\cos\theta+\cos\phi)=1$, $r^4=1$.

These give $r=1$, $\cos\theta+\cos\phi=\frac{1}{2}$, $2\cos\theta\cos\phi=0$.

Therefore either $\cos\theta=0$ or, $\cos\phi=0$.

Taking $\cos\theta=0$, we have $\cos\phi=\frac{1}{2}$, $\sin\phi=\frac{\sqrt{3}}{2}$, $\sin\theta=1$.

Taking $\cos\phi=0$, we have $\cos\theta=\frac{1}{2}$, $\sin\theta=\frac{\sqrt{3}}{2}$, $\sin\phi=1$.

Therefore the roots of the equation are $\pm i, \frac{1\pm\sqrt{3}i}{2}$.

Theorem 5.2.2. If an equation with *rational coefficients* has a surd root $\alpha+\sqrt{\beta}$, where α,β are rational and β is not a perfect square, then it has also the conjugate root $\alpha-\sqrt{\beta}$.

In other words, in an equation with rational coefficients surd roots occur in conjugate pairs.

Proof. Let $f(x)=0$ be an equation of degree n with rational coefficients and let $\alpha+\sqrt{\beta}$ be a root of $f(x)=0$. It is obvious that $n\geq 2$. Let us divide $f(x)$ by the product $(x-\alpha-\sqrt{\beta})(x-\alpha+\sqrt{\beta})$ i.e., by $(x-\alpha)^2-\beta$. Let $q(x)$ be the quotient and $r(x)$ be the remainder. Then the degree of $q(x)$ is $n-2$ and the degree of $r(x)$ is at most one.

Since $f(x)$ is a polynomial with rational coefficient and $(x-\alpha)^2-\beta$ is a quadratic with rational coefficients, $q(x)$ and $r(x)$ are both polynomials with rational coefficients. We assume $r(x)=ax+b$ where a and b are both rational.

Therefore $f(x)=[(x-\alpha)^2-\beta]q(x)+ax+b$.

Since $\alpha+\sqrt{\beta}$ is a root, $f(\alpha+\sqrt{\beta})=0$.

Therefore $a(\alpha+\sqrt{\beta})+b=0$, or, $(a\alpha+b)+a\sqrt{\beta}=0$.

But $\beta\neq 0$. Therefore $a=0, b=0$ and $f(x)=[(x-\alpha)^2-\beta]q(x)$.

$f(\alpha-\sqrt{\beta})=[(\alpha-\sqrt{\beta}-\alpha)^2-\beta]q(\alpha-\sqrt{\beta})=0$ and this proves that $\alpha-\sqrt{\beta}$ is a root.

This completes the proof.

Note. In an equation with *rational coefficients*, the number of surd roots is always even. If the degree of such an equation be odd then it must have at least one rational root. If however, the degree be even then it may not have a rational root at all.

Worked Example (continued).

4. Solve the equation $x^4 + 2x^3 - 16x^2 - 22x + 7 = 0$ which has a root $2 + \sqrt{3}$.

Since the equation contains only rational coefficients, $2 - \sqrt{3}$ is another root. Therefore $(x - 2 - \sqrt{3})(x - 2 + \sqrt{3})$, i.e., $x^2 - 4x + 1$ is a factor of the polynomial $x^4 + 2x^3 - 16x^2 - 22x + 7$.

Let $x^4 + 2x^3 - 16x^2 - 22x + 7 = (x^2 - 4x + 1)q(x)$.

Then $q(x) = x^2 + 6x + 7$. $q(x) = 0$ gives $x = -3 \pm \sqrt{2}$.

Hence the roots of the equation are $2 \pm \sqrt{3}, -3 \pm \sqrt{2}$.

Theorem 5.2.3. Let $f(x)$ be a polynomial *with real coefficients*. If α, β be two distinct real numbers such that $f(\alpha)$ and $f(\beta)$ are of opposite signs then the equation $f(x) = 0$ has at least one *real root* lying between α and β.

Proof. For the purpose of this theorem we assume x is a real variable. Since $f(x)$ is a polynomial, $f(x)$ is continuous for all real x. Since $f(\alpha)$ and $f(\beta)$ are of opposite signs, it follows from the property of a real continuous function that there is at least a real number γ between α and β such that $f(\gamma) = 0$.

That is, the equation $f(x) = 0$ has a real root γ lying between α and β.

Corollary. If $f(x)$ keeps the same sign for all real values of x, then $f(x) = 0$ has no real root and conversely.

Theorem 5.2.4. Let $f(x)$ be a polynomial *with real coefficients* and α, β be two distinct real numbers.

If $f(\alpha)$ and $f(\beta)$ are of opposite signs then the equation $f(x) = 0$ has an odd number of real roots (counting multiplicity) lying between α and β. If $f(\alpha)$ and $f(\beta)$ are of the same sign then the equation $f(x) = 0$ has either no real root or an even number of real roots (counting multiplicity) lying between α and β.

Proof. Let us assume $\alpha < \beta$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the real roots (not necessarily all distinct) of $f(x) = 0$ that lie between α and β. Then $(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_m)$ is a factor of $f(x)$.

$$\begin{aligned} \text{Let } f(x) &= (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_m)\phi(x). \\ \text{Then } f(\alpha) &= (\alpha - \alpha_1)(\alpha - \alpha_2)\ldots(\alpha - \alpha_m)\phi(\alpha) \text{ and} \\ f(\beta) &= (\beta - \alpha_1)(\beta - \alpha_2)\ldots(\beta - \alpha_m)\phi(\beta). \end{aligned}$$

$\phi(\alpha)$ and $\phi(\beta)$ must be of the same sign, because otherwise $\phi(x) = 0$ will have a real root lying between α and β and consequently that will be another real root of the equation $f(x) = 0$ lying between α and β which is contrary to our assumption.

Case 1. Let $f(\alpha)$ and $f(\beta)$ are of opposite signs.

Then $(\alpha - \alpha_1)(\alpha - \alpha_2)\ldots(\alpha - \alpha_m)$ and $(\beta - \alpha_1)(\beta - \alpha_2)\ldots(\beta - \alpha_m)$ are of opposite signs. Since $\alpha_i < \beta$ for $i = 1, 2, \ldots, m$, the product $(\beta - \alpha_1)(\beta - \alpha_2)\ldots(\beta - \alpha_m)$ is positive and therefore the product $(\alpha - \alpha_1)(\alpha - \alpha_2)\ldots(\alpha - \alpha_m)$ is negative. Since $\alpha < \alpha_i$ for $i = 1, 2, \ldots, m$ there is an odd number of factors in the product $(\alpha - \alpha_1)(\alpha - \alpha_2)\ldots(\alpha - \alpha_m)$ and this proves that m is an odd positive integer.

Case 2. Let $f(\alpha)$ and $f(\beta)$ are of the same sign.

Then $(\alpha - \alpha_1)(\alpha - \alpha_2)\ldots(\alpha - \alpha_m)$ and $(\beta - \alpha_1)(\beta - \alpha_2)\ldots(\beta - \alpha_m)$ are of the same sign. By similar arguments as in Case 1 we can prove that m is either zero or an even positive integer.

This completes the proof.

Note. Very often we shall be interested in the sign of the polynomial $f(x)$ for a sufficiently large positive value of x. Such a value will be denoted by $f(\infty)$. Similarly the value of $f(x)$ for a sufficiently large negative value of x will be denoted by $f(-\infty)$.

Worked Examples (continued).

5. Show that the equation $x^3 + 2x^2 - 2x - 1 = 0$ has one positive root and two negative roots-one lying between -3 and -1 and another lying between -1 and 0.

Let $f(x) = x^3 + 2x^2 - 2x - 1$. Then $f(\infty) > 0$ and $f(0) < 0$.

Therefore there is at least one positive root.

$f(0) < 0$ and $f(-1) > 0$.

Therefore there is at least one real root lying between -1 and 0.

$f(-3) < 0$ and $f(-1) > 0$.

Therefore there is at least one real root lying between -3 and -1.

The equation has exactly three roots. One root is positive, one root lies between -1 and 0 and one lies between -3 and -1.

6. Show that for real values of λ, the equation

$$(x + 3)(x + 1)(x - 2)(x - 4) + \lambda(x + 2)(x - 1)(x - 3) = 0$$

has all its roots real and simple.

Let $f(x) = (x + 3)(x + 1)(x - 2)(x - 4) + \lambda(x + 2)(x - 1)(x - 3)$.

Then $f(-\infty) > 0, f(-2) < 0, f(1) > 0, f(3) < 0, f(\infty) > 0$.

Therefore each of the intervals $(-\infty, -2), (-2, 1), (1, 3), (3, \infty)$ contains at least one real root of the equation $f(x) = 0$. Since the equation is of degree 4, all its roots are real and simple.

7. Show that for real values of λ, the equation
$$(x+1)(x-3)(x-5)(x-7)+\lambda(x-2)(x-4)(x-6)(x-8)=0$$
has all its roots real and simple.

Let $f(x)=(x-1)(x-3)(x-5)(x-7)+\lambda(x-2)(x-4)(x-6)(x-8)$.
Then $f(2)<0, f(4)>0, f(6)<0, f(8)>0$.
If $1+\lambda>0$ then $f(-\infty)$ and $f(\infty)$ are both positive.
In this case each of the intervals $(-\infty,2),(2,4),(4,6),(6,8)$ contains at least one real root of $f(x)=0$. So the roots are all real and simple.

If $1+\lambda<0$ then $f(-\infty)$ and $f(\infty)$ are both negative.
In this case each of the intervals $(2,4),(4,6),(6,8),(8,\infty)$ contains at least one real root of $f(x)=0$. So the roots are all real and simple.

If $1+\lambda=0$. $f(x)=0$ is a cubic equation. In this case each of the intervals $(2.4),(4.6),(6.8)$ contains a real root of $f(x)=0$.

Therefore the roots of the equation are all real and simple, whatever λ may be.

Theorem 5.2.5. Rolle's theorem.

Let $f(x)$ be a polynomial *with real coefficients*. Between two consecutive *real roots* of the equation $f(x)=0$ there is at least one real root of the equation $f'(x)=0$.

Proof. Let α,β be two consecutive real roots of the equation $f(x)=0$ with multiplicity r and s respectively. Then $(x-\alpha)^r(x-\beta)^s$ is a factor of the polynomial $f(x)$.

Let $f(x)=(x-\alpha)^r(x-\beta)^s\phi(x)$.

Then $\phi(\alpha)\neq 0$ and $\phi(\beta)\neq 0$, because otherwise the assumed multiplicity of α,β would be contradicted. Also $\phi(\alpha)$ and $\phi(\beta)$ have the same sign, because otherwise $\phi(x)=0$ would have a real root between α and β and consequently $f(x)=0$ would have a real root between α and β, which is a contradiction.

$$\begin{aligned} f'(x) &= r(x-\alpha)^{r-1}(x-\beta)^s\phi(x)+(x-\alpha)^r \\ &\qquad (x-\beta)^{s-1}\phi(x)+(x-\alpha)^r(x-\beta)^s\phi'(x) \\ &= (x-\alpha)^{r-1}(x-\beta)^{s-1}\psi(x), \text{ where} \end{aligned}$$

$\psi(x)=r(x-\beta)\phi(x)+s(x-\alpha)\phi(x)+(x-\alpha)x-\beta)\phi'(x)$.
Therefore $\psi(\alpha)=r(\alpha-\beta)\phi(\alpha), \psi(\beta)=s(\beta-\alpha)\phi(\beta)$.

Since $\phi(\alpha)$ and $\phi(\beta)$ are of the same sign, $\psi(\alpha)$ and $\psi(\beta)$ are of different signs and this shows that the equation $\psi(x)=0$ and consequently the equation $f'(x)=0$ has at least one real root between α and β.

This completes the proof.

Corollary 1. Between two consecutive real roots α',β' of the equation $f'(x)=0$ there is either no real root or at most one real root of the equation $f(x)=0$ and such a root must be simple.

Because if there exist two real roots of the equation $f(x)=0$, say α,β in between α' and β', then by the theorem there would exist a real root γ of the equation $f'(x)=0$ in between α and β and this would contradict that α',β' are consecutive.

Therefore the number of real roots of $f(x)=0$ lying between α' and β' is at most one. Such a root of $f(x)=0$ must be simple because a multiple root of the equation $f(x)=0$ must also be a root of $f'(x)=0$.

If λ' be the least and μ' be the greatest real root of the equation $f'(x)=0$, each of the intervals $(-\infty,\lambda'),(\mu',\infty)$ will contain either no real root or at most one real root of the equation $f(x)=0$.

It is evident that the interval (α',β') will contain no real root of the equation $f(x)=0$ if $f(\alpha')f(\beta')>0$ and will contain just one real root of the equation $f(x)=0$ if $f(\alpha')f(\beta')<0$.

2. If all the roots of the equation $f(x)=0$ be real and distinct, then all the roots of $f'(x)=0$ are also so. Any two consecutive roots of $f'(x)=0$ are separated by a root of $f(x)=0$.

Let the real roots of $f(x)=0$ be $\alpha_1,\alpha_2,\ldots,\alpha_n$ where $\alpha_1<\alpha_2<\ldots<\alpha_n$. There are $n-1$ intervals $(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),\ldots,(\alpha_{n-1},\alpha_n)$ each containing a real root of $f'(x)=0$, by the theorem. Since the degree of $f'(x)$ is $n-1$, these $n-1$ real roots are all the roots of $f'(x)=0$.

Theorem 5.2.6. Let $f(x)$ be a polynomial with real coefficients. If the equation $f(x)=0$ has r real roots, the number of real root of the equation $f'(x)=0$ is at least $r-1$.

Proof. Let $\alpha_1,\alpha_2,\ldots,\alpha_s$ be the distinct real roots of $f(x)=0$ with respective multiplicities $m_1,m_2,\ldots,m_s$. Then $m_1+m_2+\cdots+m_s=r$.

Let us assume $\alpha_1<\alpha_2<\ldots<\alpha_s$.

Each of the $s-1$ intervals $(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),\ldots,(\alpha_{s-1},\alpha_s)$ contains at least one real root of $f'(x)=0$, by Rolle's theorem. Therefore we have not less than $s-1$ distinct real roots of $f'(x)=0$.

Moreover, α_1 is a root of $f'(x)=0$ of multiplicity m_1-1
α_2 is a root of $f'(x)=0$ of multiplicity m_2-1
...
α_s is a root of $f'(x)=0$ of multiplicity m_s-1.

Therefore the total number of these real roots becomes

$(m_1-1)+(m_2-1)+\cdots+(m_s-1)+s-1$
$=(m_1+m_2+\cdots+m_s)-1$
$=r-1.$

Therefore the equation $f'(x)=0$ will have at least $r-1$ real roots.

This completes the proof.

Corollary. If all the roots of the equation $f(x)=0$ be real, then the equation $f'(x)=0$ cannot have an imaginary root.

Worked Examples (continued).

8. Show that the equation $(x-a)^3+(x-b)^3+(x-c)^3+(x-d)^3=0$ where a,b,c,d are positive and not all equal, has only one real root.

Let $f(x)=(x-a)^3+(x-b)^3+(x-c)^3+(x-d)^3$.

Since the equation is of degree 3 it has either only one real root or three real roots. Let us assume that the equation has more than one real roots. Let α,β be two such. Then $f'(x)=0$ has a real root between α and β.

$$\begin{aligned} f'(x) &= 3(x-a)^2+3(x-b)^2+3(x-c)^2+3(x-d)^2 \\ &= 12x^2-6(a+b+c+d)x+3(a^2+b^2+c^2+d^2). \end{aligned}$$

The discriminant of the equation $f'(x)=0$ is

$$36(a+b+c+d)^2-144(a^2+b^2+c^2+d^2).$$

Since a,b,c,d are all positive and not all equal

$$\frac{a^2+b^2+c^2+d^2}{4} > \left(\frac{a+b+c+d}{4}\right)^2$$

or, $4(a^2+b^2+c^2+d^2)>(a+b+c+d)^2$.

Therefore the discriminant of the equation $f'(x)=0$ is negative and so $f'(x)=0$ has no real root.

This proves that the equation $f(x)=0$ has only one real root.

9. Find the values of k for which the equation $x^4+4x^3-2x^2-12x+k=0$ has four real and unequal roots.

Let $f(x)=x^4+4x^3-2x^2-12x+k$.

$$\begin{aligned} \text{Then } f'(x) &= 4x^3+12x^2-4x-12 \\ &= 4(x-1)(x+3)(x+1). \end{aligned}$$

The roots of $f'(x)=0$ are $-3,-1,1$.

Since the roots of the equation $f(x)=0$ are to be all real and distinct they will be separated by the roots of the equation $f'(x)=0$.

Therefore one root is less than -3, one root lies between -3 and -1, one root lies between -1 and 1 and one root is greater than 1.

Now $f(-\infty)$ is positive, $f(-3)=-9+k, f(-1)=7+k, f(1)=-9+k, f(\infty)$ is positive.

Since only one root is less than -3 and $f(-\infty)$ is positive, therefore $f(-3)$ must be negative. Hence $k<9$... (i)

Since only one root lies between -3 and -1 and $f(-3)$ is negative, therefore $f(-1)$ must be positive. Hence $k>-7$... (ii)

Since only one root lies between -1 and 1 and $f(-1)$ is positive, therefore $f(1)$ must be negative. Hence $k<9$... (iii)

From (i),(ii) and (iii) we have $-7<k<9$.

Theorem 5.2.7. Let $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$, where $a_0,a_1,\cdots,a_n$ are integers. If p/q be a rational root of the equation $f(x)=0$ where p,q are integers prime to each other, then

(i) p is a divisor of a_n and q is a divisor of a_0,

(ii) $p-q$ is a divisor of $f(1)$ and $p+q$ is a divisor of $f(-1)$.

Proof. (i) Since p/q is a root of the equation $f(x)=0$,

$$a_0p^n+a_1p^{n-1}q+\cdots+a_{n-1}pq^{n-1}+a_nq^n=0$$

or, $a_0p^{n-1}+a_1p^{n-2}q+\cdots+a_{n-1}q^{n-1}=-\frac{a_nq^n}{p}$.

The left hand expression is an integer and therefore p is divisor of a_nq^n. Since p and q are prime to each other, p is a divisor of a_n.

Again, $-\frac{a_0p^n}{q}=a_1p^{n-1}+\cdots a_{n-1}pq^{n-2}+a_nq^{n-1}$.

The right hand expression is an integer and therefore q is divisor of a_0p^n. Since p and q are prime to each other, q is a divisor of a_0.

(ii) $$\begin{aligned} f(x+1) &= a_0(x+1)^n+a_1(x+1)^{n-1}+\cdots+a_n \\ &= A_0x^n+A_1x^{n-1}+\cdots+A_n, \text{ say} \end{aligned}$$

where $A_0=a_0, A_n=f(1)$.

Since $f(p/q)=0, \frac{p}{q}-1$ is a root of the equation $f(x+1)=0$.

Therefore $\frac{p-q}{q}$ is a root of the equation $A_0x^n+A_1x^{n-1}+\cdots+A_n=0$.

$p-q$ and q are prime to each other, since p is prime to q.

By (i) $p-q$ is a divisor of A_n, i.e., $p-q$ is a divisor of $f(1)$.

Proceeding similarly with $f(x-1)$, it can be established that $p+q$ is a divisor of $f(-1)$.

This completes the proof.

Corollary. Let $f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n$ where $p_1,p_2,\cdots,p_n$ are integers. If p be an integer root of the equation $f(x)=0$ then (i) p is a divisor of p_n and

(ii) $p-1$ is a divisor of $f(1)$ and $p+1$ is a divisor of $f(-1)$.

Worked Example (continued).

10. Find all integer roots of the equation and solve $x^4+x^3-2x^2+4x-24=0$.

Let $f(x)=x^4+x^3-2x^2+4x-24$. Then $f(1)=-20, f(-1)=-30$.

Let p be an integer root of the equation. Then p is a divisor of -24. The divisors of -24 are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$.

Since $f(1)\neq 0$ and $f(-1)\neq 0$, it follows that $p\neq 1, -1$.

$p-1$ is a divisor of -20. Therefore $p\neq -2, 4, -6, 8, -8, 12, -12, 24, -24$.

$p+1$ is a divisor of -30. Therefore $p\neq 3, 6$.

Thus the only possible values of p are $2, -3, -4$.

$f(2)=0, f(-3)=0$ and $f(-4)\neq 0$. Therefore 2 and -3 are integer roots. $(x-2)(x+3)$ is a factor of $f(x)$ and $f(x)=(x-2)(x+3)(x^2+4)$.

Therefore the roots of the equation are $2, -3, \pm 2i$.

Exercises 5A

1. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4-x^3+2x^2+x+1=0$ find the value of

(i) $(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$,

(ii) $(2\alpha+1)(2\beta+1)(2\gamma+1)(2\delta+1)$,

(iii) $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1)$,

(iv) $(\alpha^3+1)(\beta^3+1)(\gamma^3+1)(\delta^3+1)$.

2. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $x^n+nax+b=0$ prove that $(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)=n(\alpha_1^{n-1}+a)$.

3. Solve the equation, given that it has multiple roots.

(i) $x^4+2x^3+2x^2+2x+1=0$,

(ii) $x^5+3x^4+5x^3+5x^2+3x+1=0$,

(iii) $x^6+2x^5+5x^4+6x^3+7x^2+4x+2=0$.

4. If α be a double root of the equation $x^n+p_1x^{n-1}+\cdots+p_n=0$, prove that α is also a root of the equation

$$p_1x^{n-1}+2p_2x^{n-2}+3p_3x^{n-3}+\cdots+np_n=0.$$

5. Prove that the equation $1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}=0$ cannot have a multiple root.

6. Find the values of k for which the equation $x^3-9x^2+24x+k=0$ may have multiple roots and solve the equation in each case.

7. Find the values of a for which the equation $ax^3-6x^2+9x-4=0$ may have multiple roots and solve the equation in each case.

8. Prove that 1 is a multiple root of the equation $x^5-5x^3+5x^2-1=0$. Find its order and solve the equation.

9. Solve the equation

(i) $x^4-x^3+2x^2-2x+4=0$, one root being $1+i$;

(ii) $x^5-4x^4+5x^3+x^2-4x+5=0$, one root being $2+i$;

(iii) $2x^4-3x^3-3x^2-3x-1=0$, one root being $1+\sqrt{2}$;

(iv) $x^6-x^5-8x^4+2x^3+21x^2-9x-54=0$, one root being $\sqrt{2}+i$;

(v) $x^4+x^3-2x+8=0$ having a complex root of modulus $\sqrt{2}$;

(vi) $3x^4+2x^3+9x^2+4x+6=0$ having a complex root of modulus 1.

10. Form a biquadratic equation with rational coefficients two of whose roots are $\sqrt{3}\pm 2$.

11. Form a biquadratic equation with rational coefficients two of whose roots are $2i\pm 1$.

12. The equation $3x^3+5x^2+5x+3=0$ has three distinct roots of equal moduli. Solve it.

13. The equation $x^3-x^2+3x-27=0$ has three distinct roots of equal moduli. Solve it.

14. The equation $3x^4+x^3+4x^2+x+3=0$ has four distinct roots of equal moduli. Solve it.

15. The equation $x^4-2x^3+18x^2-18x+81=0$ has four distinct roots of equal moduli. Solve it.

16. Prove that the roots of the equations are all real.

(i) $\frac{1}{x+a_1}+\frac{1}{x+a_2}+\cdots+\frac{1}{x+a_n}=\frac{1}{x}$, where $a_1, a_2, \ldots, a_n$ are all positive real numbers,

(ii) $\frac{1}{x+a_1}+\frac{1}{x+a_2}+\cdots+\frac{1}{x+a_n}=\frac{1}{x}$, where $a_1, a_2, \ldots, a_n$ are all negative real numbers,

(iii) $\frac{1}{x+a_1}+\frac{1}{x+a_2}+\cdots+\frac{1}{x+a_n}=\frac{1}{x+b}$, where $a_1, a_2, \ldots, a_n$ and b are all positive real numbers and $b>a_i$ for all i,

(iv) $\frac{1}{x+a_1}+\frac{1}{x+a_2}+\cdots+\frac{1}{x+a_n}=\frac{1}{x+b}$, where $a_1, a_2, \ldots, a_n$ and b are all real and $b<a_i$ for all i,

(v) $\frac{A_1}{x+a_1}+\frac{A_2}{x+a_2}+\cdots+\frac{A_n}{x+a_n}=x+b$, where A_i, a_i, b are all real and $A_i>0$ for all i.

[**Hint.** (i) The equation is $\frac{x}{x+a_1}+\frac{x}{x+a_2}+\cdots+\frac{x}{x+a_n}=1$

or, $\frac{a_1}{x+a_1}+\frac{a_2}{x+a_2}+\cdots+\frac{a_n}{x+a_n}=(n-1)$.]

17. If a polynomial equation $f(x)=0$ with real coefficients has a complex root $(\alpha+i\beta)^p$ where α,β are real and p is a positive integer, prove that $(\alpha-i\beta)^p$ also a root of the equation.

18. If a polynomial equation $f(x)=0$ with real coefficients has a complex root $\alpha+i\beta$ of multiplicity p, prove that $\alpha-i\beta$ is also a root of the equation $f(x)=0$ of multiplicity p.

19. Prove that the roots of the equation

$$(x+4)(x+2)(x-3)+(x+3)(x+1)(x-5)=0$$

are all real and different. Separate the intervals in which the roots lie.

20. Prove that the roots of the equation

$$(2x+3)(2x+1)(x-1)(4x-7)+(x+1)(2x-1)(2x-3)=0$$

are all real and different. Separate the intervals in which the roots lie.

21. Show that the equation $x^4-14x^2+24x+k=0$ has

(i) four real and unequal roots if $-11<k<-8$,

(ii) two distinct real roots if $-8<k<117$,

(iii) no real root if $k>117$.

Discuss the cases when $k=117, k=-8$ and $k=-11$.

[**Hint.** The roots of $f'(x)=0$ are $-3,1,2$.]

22. Discuss the reality of the roots of the equation

$x^4+4x^3-12x^2-32x+k=0$ for different real values of k.

23. Find the range of values of r for which the equation

$3x^4+8x^3-6x^2-24x+r=0$ has four real and unequal roots.

24. Find all integer roots of the equation and then solve

(i) $x^4-2x^3-2x^2+3x-18=0$, (ii) $x^4+6x^3-27x-10=0$,

(iii) $x^4-8x^3+25x^2-38x+24=0$.

25. Let $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ where $a_0,a_1,\ldots,a_n$ are integers. If $f(0)$ and $f(1)$ be both odd prove that the equation $f(x)=0$ cannot have an integer root.

Hence prove that the equation $x^4+6x^3+3x^2-14x+15=0$ has no integer root.

[**Hint.** Let c be an integer root. Let c be odd. Then $c\equiv 1 \pmod 2$. This implies $f(c)\equiv f(1) \pmod 2$ and so $f(1)$ is even, since $f(c)=0$. Let c be even. Then $c\equiv 0 \pmod 2$. This implies $f(c)\equiv f(0) \pmod 2$ and so $f(0)$ is even.]

5.3. Real roots – their nature and position.

5.3.1. Limits of real roots.

Let $\alpha_1,\alpha_2,\ldots,\alpha_r$ be the real roots of an equation $f(x)=0$. The number of real roots being finite, there exist two real numbers u and l such that $l\le\alpha_i\le u$ for $i=1,2,\ldots,r$.

u is said to be *an upper limit* and l is said to be *a lower limit* of the real roots.

Let the equation $f(x)=0$ be written with the coefficient of the leading term positive.

A real number u is an upper limit of the real roots of the equation $f(x)=0$ if $f(x)>0$ for all $x\ge u$.

A real number l is a lower limit of the real roots of the equation $f(x)=0$ if it is an upper limit of the equation $f(-x)=0$.

A real number h is said to be *a lower limit of the positive roots* of $f(x)=0$, if $\frac{1}{h}$ is an upper limit of the positive roots of $f(\frac{1}{x})=0$.

Worked Examples.

1. Let $f(x)=x^4+2x^2-x-1$.

Then $f(x)>0$ if $x\ge 1$. Therefore 1 is an upper limit of the real roots.

$f(-x)=x^4+2x^2+x-1$.

1 is an upper limit of the real roots of $f(-x)=0$. Therefore -1 is a lower limit of the real roots of the equation $f(x)=0$.

$f(\frac{1}{x})=0$ gives $x^4+x^3-2x^2-1=0$.

2 is an upper limit of the positive roots of the equation $f(\frac{1}{x})=0$. Therefore $\frac{1}{2}$ is a lower limit of the positive roots of $f(x)=0$.

2. Let $f(x)=x^5-4x^3+x^2-x+1$.

Then $f(x)\ge 0$ if $x\ge 2$. Therefore 2 is an upper limit of the real roots of the equation $f(x)=0$.

$f(-x)=0$ gives $x^5-4x^3-x^2-x-1=0$.

3 is an upper limit of the roots of $f(-x)=0$. Therefore -3 is a lower limit of the real roots of the equation $f(x)=0$.

$f(\frac{1}{x})=0$ gives $x^5-x^4+x^3-4x^2+1=0$.

2 is an upper limit of the positive roots of the equation $f(\frac{1}{x})=0$. Therefore $\frac{1}{2}$ is a lower limit of the positive roots of $f(x)=0$.

Theorem 5.3.2. Let $f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n$. The positive roots of the equation $f(x)=0$ do not exceed $\sqrt[r]{p}+1$, where $-p$ is the greatest negative coefficient and p_r is the first negative coefficient of $f(x)$.

Proof. Any value of x that makes $x^n > p(x^{n-r} + x^{n-r-1} + \cdots + 1)$ will make $f(x)$ positive.

Let $x > 1$. Then

$x^n > p(x^{n-r} + x^{n-r-1} + \cdots + 1)$ will hold

if $x^n > p\frac{x^{n-r+1}-1}{x-1}$

i.e., if $x^n > p\frac{x^{n-r+1}}{x-1}$

i.e., if $x^{r-1}(x-1) > p$

i.e., if $(x-1)^r > p$

i.e., if $x > \sqrt[r]{p} + 1$.

Theorem 5.3.3. Let a be a positive number such that

$f(a) \geq 0, f'(a) \geq 0, f''(a) \geq 0, \ldots, f^n(a) \geq 0$.

Then no positive root of the equation $f(x) = 0$ exceeds a.

Proof. Let the roots of the equation $f(x) = 0$ be diminished by a by the transformation $x = y + a$.

Then $f(x) = f(y+a) = f(a) + yf'(a) + \frac{y^2}{2}f''(a) + \cdots + \frac{y^n}{n!}f^n(a)$.

Since $f(a) \geq 0, f'(a) \geq 0, \ldots, f^n(a) \geq 0$, no positive value of y can make $f(y+a) = 0$.

Therefore $f(x) = 0$ has no root greater than a. In other words, a is an upper limit of the positive roots of the equation $f(x) = 0$.

Note 1. If $f(0) \geq 0, f'(0) \geq 0, \ldots, f^n(0) \geq 0$ then the equation $f(x) = 0$ has no positive root.

Because, $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0)$ and therefore $f(x) > 0$ for each $x > 0$.

Note 2. Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ where the coefficients $a_0, a_1, \ldots, a_n$ are all non-negative. Then the equation $f(x) = 0$ has no positive root.

Because here $f(0) \geq 0, f'(0) \geq 0, \ldots, f^n(0) \geq 0$.

Practical application.

While searching for an upper limit of the real roots, the theorems discussed here can be applied to an equation. There is no general rule about the suitability or superiority of one particular method over another. Sometimes one of them gives a closer limit, sometime another.

Sometimes a better approximation is achieved by grouping the terms of the equation in a suitable manner so that the sum of the terms in each group ≥ 0 for $x \geq h$. Then obviously h becomes an upper limit of the real roots.

Worked Examples (continued).

3. Find an upper limit of the real roots of the equation $x^4 - 2x^3 + 3x^2 - 2x + 2 = 0$.

Let $f(x) = x^4 - 2x^3 + 3x^2 - 2x + 2$.

The equation can be arranged as $x^3(x-2) + 3x(x - \frac{2}{3}) + 2 = 0$.

$f(x) > 0$ for $x \geq 2$. Therefore 2 is an upper limit of the real roots.

By the theorem **5.3.2,** $2 + 1$ is an upper limit.

Let us apply the theorem **5.3.3.**

$f'(x) = 4x^3 - 6x^2 + 6x - 2 = 2(2x^3 - 3x^2 + 3x - 1)$,

$f''(x) = 12x^2 - 12x + 6 = 6(2x^2 - 2x + 1)$,

$f'''(x) = 24x - 12 = 12(2x - 1)$.

$f'''(1) > 0, f''(1) > 0, f'(1) > 0, f(1) > 0$.

Therfore 1 is an upper limit of the real roots.

Note. The theorem 5.3.3 gives the closest limit.

4. Find an upper limit of the real roots of the equation $x^4 - x^3 - 2x^2 - 4x + 1 = 0$.

Multiplying the equation by 3 the equation can be arranged as

$(x^4 - 3x^3) + (x^4 - 6x^2) + (x^4 - 12x) + 3 = 0$

or, $x^3(x-3) + x^2(x^2 - 6) + x(x^3 - 12) + 3 = 0$.

Each group ≥ 0 for $x \geq 3$. Therefore 3 is an upper limit of the roots of the equation.

By the theorem **5.3.2,** $4 + 1$ is an upper limit.

Let us apply the theorem **5.3.3.**

Let $f(x) = x^4 - x^3 - 2x^2 - 4x + 1$.

Then $f'(x) = 4x^3 - 3x^2 - 4x - 4$,

$f''(x) = 12x^2 - 6x - 4 = 2(6x^2 - 3x - 2)$,

$f'''(x) = 24x - 6 = 6(4x - 1)$.

$f(x) \geq 0$ for $x \geq 3$, $f'(x) \geq 0$ for $x \geq 2$, $f''(x) \geq 0$ for $x \geq 1$, $f'''(x) \geq 0$ for $x \geq 1$.

Hence 3 is an upper limit.

5. Find an upper limit of the real roots of the equation $x^4 - 3x^3 - 2x^2 + 7x + 3 = 0$.

The equation can be arranged as $x^2(x^2 - 3x - 2) + (7x + 3) = 0$.

$x \geq 4$ makes each group positive.

Therefore 4 is an upper limit of the roots.

By the theorem **5.3.2,** $3 + 1$ is an upper limit.

Let us apply the theorem **5.3.3.**

$$\begin{aligned} \text{Let } f(x) &= x^4 - 3x^3 - 2x^2 + 7x + 3. \\ \text{Then } f'(x) &= 4x^3 - 9x^2 - 4x + 7, \\ f''(x) &= 12x^2 - 18x - 4 = 2(6x^2 - 9x - 2), \\ f'''(x) &= 24x - 18 = 6(4x - 3). \end{aligned}$$

$f(x) \geq 0$ for $x \geq 3$, $f'(x) \geq 0$ for $x \geq 3$, $f''(x) \geq 0$ for $x \geq 2$, $f'''(x) \geq 0$ for $x \geq 1$.

Hence 3 is an upper limit.

5.3.4. Descartes' rule of signs.

In a sequence of real numbers $a_0, a_1, \ldots, a_n$, none of which is zero, the signs of two consecutive elements may be same or different. When same sign occurs we say that the elements show a *continuation of signs*; when the signs are different we say that the elements show a *variation of signs*.

For example, in the sequence $1, 3, -5, -7, 9, -4, 10, 2$ there are 3 continuations and 4 variations of signs.

If some of the elements of a sequence be zero, we ignore their presence in the sequence and count the number of continuations and variations of signs.

For example, in the sequence $1, 3, -2, 0, -3, 0, 4, 0, 0, 7$ there are 3 continuations and 2 variations of signs.

We state here without proof a celebrated theorem of **Descartes**, known as "Descartes' rule of signs".

Statement of the rule.

The number of positive roots of an equation $f(x) = 0$ with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of $f(x)$ and if less, it is less by an even number.

This says that if v be the number of variations of signs and r be the number of positive roots, then $v = r + 2h$ where h is a non-negative integer.

Deductions.

1. The number of negative roots of an equation $f(x) = 0$ with real coefficients does not exceed the number of variations of signs in the sequence of coefficients of $f(-x)$ and if less, it is less by an even number.

Proof. Let r' be the number of negative roots of the equation $f(x) = 0$.

Then r' is the number of positive roots the equation $f(-x) = 0$. Let v' be the number of variations of signs in the sequence of coefficients of $f(-x)$. By the rule, $v' = r' + 2h$ where h is a non-negative integer.

2. If $f(x) = 0$ be an equation of degree n with real coefficients having no zero root and v, v' are respectively the number of variations of signs in the sequence of coefficients of $f(x)$ and $f(-x)$ such that $v + v' < n$, then the equation $f(x) = 0$ has at least $n - (v + v')$ complex roots.

Proof. Let r be the number of positive roots and r' be that of negative roots of the equation $f(x) = 0$. Then by the rule, $v = r + 2h$ and $v' = r' + 2h'$ where h, h' are non-negative integers.

Therefore $r \leq v, r' \leq v'$ and $n - (r + r') \geq n - (v + v')$.

But $n - (r + r')$ is the number of complex roots of the equation $f(x) = 0$ and the assertion is established.

3. If all the roots of the equation $f(x) = 0$ be non-zero real and v, v' are respectively the number of variations of signs in the sequence of coefficients of $f(x)$ and $f(-x)$ then the equation $f(x) = 0$ has v positive roots and v' negative roots.

Proof. Let r, r' be respectively the number of positive and negative roots of $f(x) = 0$ and let n be its degree. Then $r + r' = n$. Again the minimum number of complex roots of the equation is $n - (v + v')$, by deduction 2.

Since all the roots are real, $n - (v + v') = 0$. Therefore $r + r' = v + v' = n$.

By the rule, $v \geq r$ and $v' \geq r'$. But $v > r$ and $v + v' = r + r' \Rightarrow v' < r'$, a contradiction.

Therefore $v = r$ and consequently $v' = r'$.

Worked Examples (continued).

6. Apply Descartes' rule of signs to examine the nature of the roots of the equations

(i) $x^4 + 2x^2 + 3x - 1 = 0$,

(ii) $x^6 + x^4 + x^2 + x + 3 = 0$.

(i) Let $f(x) = x^4 + 2x^2 + 3x - 1$.

Then $f(-x) = x^4 + 2x^2 - 3x - 1$.

The signs in the sequence of coefficients of $f(x)$ are $+ + + -$.

There is only one variation of signs and therefore the number of positive roots of $f(x) = 0$ is exactly 1.

The signs in the sequence of coefficients of $f(-x)$ are $+ + - -$.

There is only one variation of signs and therefore the number of negative roots of $f(x) = 0$ is exactly 1.

The equation has no zero root. Therefore the number of real roots is 2. The equation being of degree 4 has 4 roots. Consequently, the number of complex roots of the equation is 2.

(ii) Let $f(x) = x^6 + x^4 + x^2 + x + 3$.
Then $f(-x) = x^6 + x^4 + x^2 - x + 3$.

The signs in the sequence of coefficients of $f(x)$ are $+ + + + + +$.
As there is no variation of signs the equation has no positive root.

The signs in the sequence of coefficients of $f(-x)$ are $+ + + - + +$.
There are 2 variations of signs. Therefore the number of negative roots of the equation $f(x) = 0$ is either 2 or 0.

The equation has no zero root. Therefore the number of real roots is either 2 or 0.

The equation being of degree 6 has 6 roots. Consequently, the number of complex roots of the equation is either 4 or 6.

7. Let $f(x)$ be a complete polynomial, with real coefficients and v, v' are respectively the number of variations and continuations of signs in the sequence of its coefficients. If all roots of $f(x) = 0$ are real prove that the equation has v positive roots and v' negative roots.

[A polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_n$ is said to be a *complete polynomial* if no a_i is zero.]

Since $f(x)$ is a complete polynomial, if two consecutive elements in the sequence of coefficients of $f(x)$ show a variation of signs then the two corresponding elements in the sequence of coefficients of $f(-x)$ show a continuation of signs and vice-versa.

Therefore v' is the number of variations of signs in the sequence of coefficients of $f(-x)$.

Since $f(x)$ is a complete polynomial, 0 is not a root of $f(x) = 0$. By deduction 3 of the rule, the desired result is derived.

5.3.5. Sturm's method for location of roots.

Descartes' rule of signs does not give the exact number of real roots of an equation. Sturm's method which will be discussed here gives the exact number as well as the position of the real roots of an equation with real coefficients.

Sturm's functions. Let $f(x)$ be a polynomial with real coefficients and $f_1(x)$ be its first derivative. Let the operation of finding the h.c.f. of $f(x)$ and $f_1(x)$ be performed with the following modification.

"The sign of each remainder is to be changed before it is used as the next divisor and the sign of the last remainder is also to be changed."

The modified remainders (i.e., the remainders with their signs changed) are denoted by $f_2(x), f_3(x), \ldots, f_r(x)$.

$f(x), f_1(x), \ldots, f_r(x)$ are called *Sturm's functions* and $f_1(x), f_2(x), \ldots, f_r(x)$ are called *auxiliary functions.*

In the usual process of finding the h.c.f. of two polynomials we can multiply (or divide) any remainder by a constant, positive or negative but in this modified process of forming Strum's auxiliary functions it is essential that such multipliers should be positive.

If $q_1(x), q_2(x), \ldots, q_{r-1}(x)$ be the quotients in the successive steps, the Sturm's functions $f, f_1, f_2, \ldots, f_r$ are connected by the equations

$$\begin{aligned} f &= q_1 f_1 - f_2 \\ f_1 &= q_2 f_2 - f_3 \\ &\cdots \quad \cdots \quad \cdots \\ f_{r-2} &= q_{r-1} f_{r-1} - f_r. \end{aligned}$$

It is to be observed that the last remainder $-f_r(x)$ is a constant if $f(x)$ has no multiple root. But if $f(x)$ has multiple roots then the h.c.f. of $f(x)$ and $f_1(x)$ is a polynomial, say $p(x)$. In this case all the Sturm's functions are divisible by $p(x)$ and the remainder $-f_r(x) = p(x)\phi_r(x)$, where $\phi_r(x)$ is a constant.

We state below without proof *Sturm's theorem* for locating the position of real roots of an equation.

Sturm's theorem.

I. All roots unequal.

Let $f(x)$ be a polynomial with real coefficients and a, b are real numbers $(a < b)$. The number of real roots of the equation $f(x) = 0$ lying between a and b is equal to the excess of the number of changes of signs in the sequence of Sturm's functions $f(x), f_1(x), \ldots, f_r(x)$ when $x = a$ over the number of changes of signs in the sequence when $x = b$.

II. Equal roots.

Let $f(x)$ be a polynomial with real coefficients and a, b are real num-

bers ($a < b$). The number of real roots of the equation $f(x) = 0$ lying between a and b (a multiple root, if there be any, being counted once only) is equal to the excess of the number of changes of signs in the sequence of Sturm's functions $f(x), f_1(x), \ldots, f_r(x)$ when $x = a$ over the number of changes of signs in the sequence when $x = b$.

Worked Examples (continued).

8. Find the number and position of the real roots of the equation $x^3 - 3x + 1 = 0$.

Here $f(x) = x^3 - 3x + 1$,
$f_1(x) = x^2 - 1$, (removing the factor 3)
$f_2(x) = 2x - 1$. $f_3(x) = 3$.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	changes of sign
$(-\infty)$	$-$	$+$	$-$	$+$	3
0	$+$	$-$	$-$	$+$	2
∞	$+$	$+$	$+$	$+$	0

The equation has 3 real roots, one negative and two positive.

Location of roots.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	changes of sign
-2	$-$	$+$	$-$	$+$	3
-1	$+$	0	$-$	$+$	2
0	$+$	$-$	$-$	$+$	2
1	$-$	0	$+$	$+$	1
2	$+$	$+$	$+$	$+$	0

One root lies between -2 and -1; one lies between 0 and 1; and one lies between 1 and 2.

9. Find the number and position of the real roots of the equation $x^4 - 6x^3 + 10x^2 - 6x + 1 = 0$.

Here $f(x) = x^4 - 6x^3 + 10x^2 - 6x + 1$,
$f_1(x) = 2x^3 - 9x^2 + 10x - 3$, (removing the factor 2)
$f_2(x) = 7x^2 - 12x + 5$, $f_3(x) = x - 1$.

$f_3(x)$ divides $f_2(x)$ without remainder. This establishes that the given equation has multiple roots. 1 is a double root.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	changes of sign
$(-\infty)$	$+$	$-$	$+$	$-$	3
0	$+$	$-$	$+$	$-$	3
∞	$+$	$+$	$+$	$+$	0

The equation has 3 distinct real roots, but one of them is a double root. Therefore all the roots of the equation are real and all of them are positive.

Location of roots.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	changes of sign
0	$+$	$-$	$+$	$-$	3
$\frac{1}{2}$	$-$	0	$+$	$-$	2
2	$-$	$-$	$+$	$+$	1
3	$-$	0	$+$	$+$	1
4	$+$	$+$	$+$	$+$	0

One root lies between 0 and $\frac{1}{2}$; one root (the double root 1) lies between $\frac{1}{2}$ and 2; and one lies between 3 and 4.

Remarks. The calculation of Sturm's functions becomes very often laborious. But labour may be saved by the following considerations.

1. If there is no multiple root the last function $f_r(x)$ is a constant and its sign only is required. Let α be a root of $f_{r-1}(x) = 0$ then $f_{r-2}(\alpha)$ and $f_r(\alpha)$ are of opposite signs, since $f_{r-2}(\alpha) = q_{r-1} f_{r-1}(\alpha) - f_r(\alpha)$. Therefore the sign of $f_r(x)$ can be ascertained from the calculation of $f_{r-2}(\alpha)$. Thus the labour of actual calculation of the constant $f_r(x)$ can be saved.

2. If at any stage we obtain a function $f_s(x)$ such that all of its roots are complex, then the h.c.f. process need not be continued further and the determination and location of real roots will be possible from the set of functions $f(x), f_1(x), \ldots, f_s(x)$, because $f_s(x)$ retains the same sign for all values of x and no alteration in the number of changes of sign can take place in the sequence of functions beyond $f_s(x)$.

Condition that all roots may be real and distinct.

Let $f(x)$ be a polynomial of degree n with *leading coefficient positive*. In order to ensure the existence of n *distinct real roots*, n changes of signs must be lost in the sequence of Sturm's functions as x changes from $-\infty$ to ∞. Therefore $n+1$ functions must be present in the sequence and the leading coefficients of all these functions must be positive.

Worked Examples (continued).

10. Find the nature of real roots of the equation

$$x^5 - 5x + 2 = 0.$$

$f(x) = x^5 - 5x + 2$, $f_1(x) = x^4 - 1$, (removing the factor 5)
$f_2(x) = 2x - 1$ (removing the factor 2).
Therefore $f_3(x)$ is a constant.

$f_2(x) = 0$ when $x = \frac{1}{2}$. Therefore $f_1(\frac{1}{2})$ and $f_3(\frac{1}{2})$ are of opposite signs. $f_1(\frac{1}{2}) < 0$. Therefore $f_3(\frac{1}{2}) > 0$ and so $f_3(x) > 0$ for all real x.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	changes of sign
$(-\infty)$	$-$	$+$	$-$	$+$	3
0	$+$	$-$	$-$	$+$	2
∞	$+$	$+$	$+$	$+$	0

The equation has 3 real roots, one negative and two positive.

11. Find the number of real roots of the equation

$$x^4 + 4x^3 - x^2 - 2x - 5 = 0.$$

$f(x) = x^4 + 4x^3 - x^2 - 2x - 5$, $f_1(x) = 2x^3 + 6x^2 - x - 1$,
$f_2(x) = 7x^2 + 2x + 9$.

The roots of $f_2(x) = 0$ are all imaginary. Therefore $f_2(x) > 0$ for all real x.

The remaining Sturm's functions need not be calculated.

	$f(x)$	$f_1(x)$	$f_2(x)$	changes of sign
$(-\infty)$	$+$	$-$	$+$	2
0	$-$	$-$	$+$	1
∞	$+$	$+$	$+$	0

The equation has two real roots, one positive and one negative.

12. Find the condition that the roots of the equation $x^3 + 3Hx + G = 0$ may have three real and distinct roots.

The Sturm's functions are

$f(x) = x^3 + 3Hx + G$, $f_1(x) = x^2 + H$,
$f_2(x) = -2Hx - G$, $f_3(x) = -(G^2 + 4H^3)$.

In order that the roots may be all real and distinct,

(i) there sould be four Sturm's functions and

(ii) the leading coefficient of each function should be positive.

Hence the conditions are $-2H > 0$ and $-(G^2 + 4H^3) > 0$,
i.e., $H < 0$ and $G^2 + 4H^3 < 0$.

These two conditions can be expressed as the single condition $G^2 + H^3 < 0$, because the condition $G^2 + 4H^3 < 0$ implies $H < 0$.

Exercises 5B

1. Find an upper limit of the real roots of the equation

(i) $x^4 + 4x^3 - 11x^2 - 9x - 50 = 0$,

(ii) $x^4 - 5x^3 + 40x^2 - 8x + 24 = 0$,

(iii) $x^5 + x^4 - 6x^3 - 8x^2 - 15x - 10 = 0$.

[Hint. (i) Use theorem 5.3.3; (ii) $x^2(x^2 - 5x + 20) + 20(x^2 - \frac{2}{5}x + \frac{6}{5}) = 0$;
(iii) $x(x^4 - 6x^2 - 15) + (x^4 - 8x - 10) = 0$.]

2. Calculate Sturm's functions and locate the position of the real roots of the equation

(i) $x^3 - 3x - 1 = 0$, (ii) $x^3 - 7x + 7 = 0$,

(iii) $x^4 - x^2 - 2x - 5 = 0$, (iv) $x^5 - 5x + 5 = 0$.

3. Show that the equations have no real root.

(i) $x^4 - x + 3 = 0$, (ii) $x^6 - x + 6 = 0$,

(iii) $x^4 + 3x^3 - x^2 - 3x + 11 = 0$.

4. If a and b are positive prove that the equation $x^5 - 5ax + 4b = 0$ has three real roots or only one according as $a^5 >$ or $< b^4$.

5. Apply Descartes' rule of signs to find the nature of the roots of the equation

(i) $x^4 + 2x^2 + 3x - 1 = 0$, (ii) $x^8 + 1 = 0$,

(iii) $x^{10} - 1 = 0$, (iv) $x^7 + x^5 - x^3 = 0$.

6. Apply Descartes' rule of signs to ascertain the minimum number of complex roots of the equation

(i) $x^6 - 3x^2 - 2x - 3 = 0$, (ii) $x^7 - 3x^3 - x + 1 = 0$,

(iii) $x^7 - 3x^3 + x^2 = 0$.

7. If $f(x) = 2x^3 + 7x^2 - 2x - 3$ express $f(x - 1)$ as a polynomial in x. Apply Descartes' rule of signs to both the equations $f(x) = 0$ and $f(-x) = 0$ to determine the exact number of positive and negative roots of $f(x) = 0$.

8. (a) Use Sturm's functions to show that the roots of the equations are all real and distinct.

(i) $x^3 + 3x^2 - 3 = 0$, (ii) $x^3 + 3x^2 - 9x - 3 = 0$,

(iii) $x^4 + 5x^3 - 13x + 5 = 0$, (iv) $x^4 - 12x^2 + 4 = 0$,

(v) $x^4 + 4x^3 - x^2 - 10x + 3 = 0$.

(b) Use Descartes' rule of signs to find the number of positive and negative roots of the equations in **(a)**.

5.4. Relation between roots and coefficients.

Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ be a polynomial of degree n with coefficients real or complex. Then $a_0 \neq 0$.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of the equation $f(x) = 0$. Then

$a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$.

$= a_0[x^n - \Sigma\alpha_1 x^{n-1} + \Sigma\alpha_1\alpha_2 x^{n-2} - \dots + (-1)^n(\alpha_1\alpha_2\dots\alpha_n)]$, where

$\Sigma\alpha_1$ = sum of the roots,

$\Sigma\alpha_1\alpha_2$ = sum of the products of the roots taken two at a time,

... ...

$\Sigma\alpha_1\alpha_2\dots\alpha_r$ = sum of the products of the roots taken r at a time.

From the equality of polynomials it follows that

$$a_1 = a_0(-\Sigma\alpha_1),$$
$$a_2 = a_0(\Sigma\alpha_1\alpha_2),$$
$$a_3 = a_0(-\Sigma\alpha_1\alpha_2\alpha_3),$$
$$\dots \quad \dots$$
$$a_n = a_0(-1)^n\alpha_1\alpha_2\dots\alpha_n.$$

Therefore $\Sigma\alpha_1 = -\frac{a_1}{a_0}$,

$\Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}$,

$\Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$,

...,...,...

$\alpha_1\alpha_2\dots\alpha_n = (-1)^n\frac{a_n}{a_0}$.

Particular cases.

1. If α, β, γ be the roots of the cubic equation $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ then

$$\Sigma\alpha = -\frac{a_1}{a_0}, \quad \Sigma\alpha\beta = \frac{a_2}{a_0}, \quad \alpha\beta\gamma = -\frac{a_3}{a_0}.$$

2. If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic equation $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ then

$$\Sigma\alpha = -\frac{a_1}{a_0}, \quad \Sigma\alpha\beta = \frac{a_2}{a_0}, \quad \Sigma\alpha\beta\gamma = -\frac{a_3}{a_0}, \quad \alpha\beta\gamma\delta = \frac{a_4}{a_0}.$$

Worked Examples.

1. Solve the equation $2x^3 - x^2 - 18x + 9 = 0$ if two of the roots are equal in magnitude but opposite in sign.

Let the roots be α, β, γ and $\alpha = -\beta$.

Then $\alpha + \beta + \gamma = \frac{1}{2}$ ... (i)

$\alpha\beta + \beta\gamma + \gamma\alpha = -9$ ... (ii)

$\alpha\beta\gamma = -\frac{9}{2}$ ... (iii)

Since $\alpha + \beta = 0$, from (i) $\gamma = \frac{1}{2}$ and from (iii) $\alpha^2 = 9$.

Therefore $\alpha = \pm 3$. Hence the roots are $3, -3, \frac{1}{2}$.

2. Solve the equation $16x^4 - 64x^3 + 56x^2 + 16x - 15 = 0$ whose roots are in arithmetic progression.

Let the roots be $\alpha - 3\delta, \alpha - \delta, \alpha + \delta, \alpha + 3\delta$. Then

$$4\alpha = \frac{64}{16} \quad \dots \text{(i)}$$

$$\{(\alpha-3\delta)+(\alpha+3\delta)\}\{(\alpha-\delta)+(\alpha+\delta)\} + (\alpha^2-9\delta^2) + (\alpha^2-\delta^2) = \frac{56}{16} \quad \dots \text{(ii)}$$

$$\{(\alpha-3\delta)(\alpha+3\delta)\}\{(\alpha-\delta)+(\alpha+\delta)\} + (\alpha-\delta)(\alpha+\delta)\{(\alpha-3\delta)+(\alpha+3\delta)\} = -\frac{16}{16} \quad \dots \text{(iii)}$$

$$(\alpha^2-9\delta^2)(\alpha^2-\delta^2) = -\frac{15}{16} \quad \dots \text{(iv)}$$

From (i) $\alpha = 1$ and from (ii) $6\alpha^2 - 10\delta^2 = \frac{7}{2}$ or, $\delta = \pm\frac{1}{2}$.

Therefore the roots are $1 - \frac{3}{2}, 1 - \frac{1}{2}, 1 + \frac{1}{2}, 1 + \frac{3}{2}$, i.e., $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$.

3. Solve the equation $2x^4 - 5x^3 - 15x^2 + 10x + 8 = 0$, the roots being in geometric progression.

First we observe that if four numbers a, b, c, d be in geometric progression then $ad = bc$.

Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation. Then

$\alpha + \beta + \gamma + \delta = \frac{5}{2}$ ... (i)

$(\alpha+\delta)(\beta+\gamma) + \alpha\delta + \beta\gamma = -\frac{15}{2}$ ... (ii)

$\alpha\delta(\beta+\gamma) + \beta\gamma(\alpha+\delta) = -5$ ... (iii)

$\alpha\delta.\beta\gamma = 4$ ... (iv)

$\alpha\delta = \beta\gamma$ ... (v)

From (i),(iii) and (v) we have $\alpha\delta = \beta\gamma = -2$ ... (vi)

From (ii) $(\alpha+\delta)(\beta+\gamma) = -\frac{7}{2}$ ... (vii)

From (i) and (vii) it follows that $\alpha + \delta, \beta + \gamma$ are the roots of the equation $t^2 - \frac{5}{2}t - \frac{7}{2} = 0$. Therefore $t = -1, \frac{7}{2}$... (viii)

From (vi) and (viii) it follows that one of the pairs (α, δ) and (β, γ) are the roots of the equation $x^2 + x - 2 = 0$ and the other pair are the roots of the equation $y^2 - \frac{7}{2}y - 2 = 0$.

Solving, we have $x = 1, -2; y = -\frac{1}{2}, 4$.

Hence the roots of the given equation are $-\frac{1}{2}, 1, -2, 4$.

4. If α be a multiple root of order 3 of the equation $x^4 + bx^2 + cx + d = 0$, $(d \neq 0)$, show that $\alpha = -\frac{8d}{3c}$.

Since $d \neq 0$, no root of the equation is 0. Let the roots be $\alpha, \alpha, \alpha, \beta$. Then $\alpha \neq 0, \beta \neq 0$.

We have
$$\begin{aligned} 3\alpha + \beta &= 0 \quad \ldots \text{(i)} \\ 3\alpha^2 + 3\alpha\beta &= b \quad \ldots \text{(ii)} \\ \alpha^3 + 3\alpha^2\beta &= -c \quad \ldots \text{(iii)} \\ \alpha^3\beta &= d \quad \ldots \text{(iv)} \end{aligned}$$

From (i) $\beta = -3\alpha$; from (iii) $8\alpha^3 = c$; from (iv) $3\alpha^4 = -d$.

Since $\alpha \neq 0, c \neq 0$, we have $\frac{3\alpha^4}{8\alpha^3} = -\frac{d}{c}$ or, $\alpha = -\frac{8d}{3c}$.

5. Find the relation among p, q, r, s so that the product of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ is unity.

Let $\alpha, \beta, \gamma, \delta$ be the roots and $\alpha\beta = 1$. Then
$$\begin{aligned} \alpha + \beta + \gamma + \delta &= -p \quad \ldots \text{(i)} \\ (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta &= q \quad \ldots \text{(ii)} \\ \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) &= -r \quad \ldots \text{(iii)} \\ \alpha\beta\gamma\delta &= s \quad \ldots \text{(iv)} \end{aligned}$$

From (iv) $\gamma\delta = s$ and from (iii) $(\gamma + \delta) + s(\alpha + \beta) = -r \quad \ldots$ (v)

From (i) and (v) $\alpha + \beta = \frac{r-p}{1-s}$.

From (i) $\gamma + \delta = -p - \frac{r-p}{1-s} = \frac{ps-r}{1-s}$.

From (ii) $\left(\frac{r-p}{1-s}\right)\left(\frac{ps-r}{1-s}\right) + 1 + s = q$

or, $(r-p)(ps-r) = (1-s)^2(q-s-1)$.

5.5. Symmetric functions of roots.

A function f of two or more variables is said to be a *symmetric function* if f remains unaltered by an interchange of any two of its variables.

For example, $f(x, y, z) = x^2y^2 + y^2z^2 + z^2x^2$ is a symmetric function of x, y, z. $f(x, y, z) = xy + yz$ is not symmetric in x, y, z, because f does not remain unaltered if x and y are interchanged.

A symmetric function of the roots of an equation is an expression that involves all the roots alike and the expression remains unaltered if any two of the roots be interchanged.

For example, if α, β, γ be the roots of a cubic equation, $\alpha^2 + \beta^2 + \gamma^2$ is a symmetric function of the roots, while $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$ is not a symmetric function.

A symmetric function which is the sum of a number of terms of the same type is represented by any one of its terms with a Σ (sigma) before it.

For example, the symmetric function $\alpha^2 + \beta^2 + \gamma^2$ is represented by $\Sigma\alpha^2$, $\alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta$ is represented by $\Sigma\alpha^2\beta\gamma$.

For three variables α, β, γ, $\Sigma\alpha^2\beta^2$ stands for $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$, and for four variables $\alpha, \beta, \gamma, \delta$, $\Sigma\alpha^2\beta^2$ stands for $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\delta^2 + \beta^2\delta^2 + \gamma^2\delta^2$.

Worked Examples.

1. If α, β, γ be the roots of the cubic equation $x^3 + px^2 + qx + r = 0$, find the value of

(i) $\Sigma\alpha^2$, (ii) $\Sigma\alpha^2\beta$, (iii) $\Sigma\alpha^3$, (iv) $\Sigma\alpha^2\beta^2$, (v) $\Sigma\frac{1}{\alpha}$, (vi) $\Sigma\frac{1}{\alpha\beta}$, (vii) $\Sigma\frac{1}{\alpha^2}$.

Since α, β, γ are the roots, $\Sigma\alpha = -p$, $\Sigma\alpha\beta = q$, $\alpha\beta\gamma = -r$.

(i) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = p^2 - 2q$,

(ii) $\Sigma\alpha^2\beta = \Sigma\alpha.\Sigma\alpha\beta - 3\alpha\beta\gamma = -pq + 3r$,

(iii) $\Sigma\alpha^3 = \Sigma\alpha^2.\Sigma\alpha - \Sigma\alpha^2\beta = (p^2 - 2q)(-p) - (-pq + 3r)$
$= -p^3 + 3pq - 3r$,

(iv) $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\Sigma\alpha(\alpha\beta\gamma) = q^2 - 2pr$,

(v) $\Sigma\frac{1}{\alpha} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = -\frac{q}{r}$,

(vi) $\Sigma\frac{1}{\alpha\beta} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{p}{r}$,

(vii) $\Sigma\frac{1}{\alpha^2} = (\Sigma\frac{1}{\alpha})^2 - 2\Sigma\frac{1}{\alpha\beta} = \frac{q^2}{r^2} - \frac{2p}{r} = \frac{q^2-2pr}{r^2}$.

Note. It is assumed in (v),(vi) and (vii) that none of the roots is zero, i.e., $r \neq 0$.

2. If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$, find the value of

(i) $\Sigma\alpha^2$, (ii) $\Sigma\alpha^2\beta$, (iii) $\Sigma\alpha^2\beta\gamma$, (iv) $\Sigma\alpha^2\beta^2$, (v) $\Sigma\alpha^2\beta^2\gamma^2$, (vi) $\Sigma\frac{1}{\alpha}$, (vii) $\Sigma\frac{1}{\alpha\beta}$, (viii) $\Sigma\frac{1}{\alpha^2}$.

Since α, β, γ are the roots,
$$\begin{aligned} \Sigma\alpha &= -p \\ \Sigma\alpha\beta &= q \\ \Sigma\alpha\beta\gamma &= -r \\ \alpha\beta\gamma\delta &= s. \end{aligned}$$

(i) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = p^2 - 2q$,

(ii) $\Sigma\alpha^2\beta = \Sigma\alpha.\Sigma\alpha\beta - 3\Sigma\alpha\beta\gamma = -pq + 3r$,

(iii) $\Sigma\alpha^2\beta\gamma = \Sigma\alpha.\Sigma\alpha\beta\gamma - 4\alpha\beta\gamma\delta = pr - 4s$,

(iv) $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\Sigma\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta = q^2 - 2pr + 2s$,

(v) $\Sigma\alpha^2\beta^2\gamma^2 = (\Sigma\alpha\beta\gamma)^2 - 2\Sigma\alpha^2\beta^2\gamma\delta = r^2 - 2qs$,

(vi) $\Sigma\frac{1}{\alpha} = \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta} = -\frac{r}{s}$,

(vii) $\Sigma\frac{1}{\alpha\beta} = \frac{\Sigma\alpha\beta}{\alpha\beta\gamma\delta} = \frac{q}{s}$,

(viii) $\Sigma\frac{1}{\alpha^2} = (\Sigma\frac{1}{\alpha})^2 - 2\Sigma\frac{1}{\alpha\beta} = \frac{r^2}{s^2} - \frac{q}{s} = \frac{r^2-2qs}{s}$.

Note. It is assumed in (vi) (vii) and (viii) that none of the roots is zero, i.e., $s \neq 0$.

3. If α, β, γ be the roots of the equation $x^3 + qx + r = 0$, find the value of (i) $\Sigma\alpha^5$, (ii) $\Sigma\frac{1}{\alpha^2-\beta\gamma}$.

(i) Since α, β, γ are the roots, $\Sigma\alpha = 0, \Sigma\alpha\beta = q, \alpha\beta\gamma = -r$.

Also $\alpha^3 + q\alpha + r = 0$

$\beta^3 + q\beta + r = 0$

$\gamma^3 + q\gamma + r = 0$.

Therefore $\Sigma\alpha^3 + q\Sigma\alpha + 3r = 0$

or, $\Sigma\alpha^3 = -3r$.

Again, $\alpha^5 + q\alpha^3 + r\alpha^2 = 0$

$\beta^5 + q\beta^3 + r\beta^2 = 0$

$\gamma^5 + q\gamma^3 + r\gamma^2 = 0$.

Hence $\Sigma\alpha^5 = -q\Sigma\alpha^3 - r\Sigma\alpha^2$

$= 3qr - r(-2q)$, since $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= 5qr$.

(ii) $\alpha^2 - \beta\gamma = \alpha.\alpha - \beta\gamma$

$= -\alpha(\beta + \gamma) - \beta\gamma$, since $\alpha + \beta + \gamma = 0$

$= -(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= -q$.

Therefore $\Sigma\frac{1}{\alpha^2-\beta\gamma} = -\frac{3}{q}$.

4. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0 \ (p_n \neq 0),$$

find the value of (i) $\Sigma\frac{1}{\alpha_1}$, (ii) $\Sigma\frac{\alpha_1}{\alpha_2}$, (iii) $\Sigma\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}$.

Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots,

$\Sigma\alpha_1 = -p_1, \Sigma\alpha_1\alpha_2 = p_2, \ldots, \alpha_1\alpha_2\ldots\alpha_n = (-1)^n p_n$.

(i) $\Sigma\frac{1}{\alpha_1} = \frac{\Sigma\alpha_1\alpha_2\ldots\alpha_{n-1}}{\alpha_1\alpha_2\ldots\alpha_n} = \frac{(-1)^{n-1}p_{n-1}}{(-1)^n p_n} = -\frac{p_{n-1}}{p_n}$,

(ii) $\Sigma\frac{\alpha_1}{\alpha_2} = \Sigma\alpha_1.\Sigma\frac{1}{\alpha_1} - n = \frac{p_1p_{n-1}}{p_n} - n$,

(iii) $\Sigma\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2} = \Sigma(\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1})$

$= \alpha_1(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n}) - 1$

$+\alpha_2(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n}) - 1$

$+ \cdots \quad \cdots$

$+\alpha_n(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n}) - 1$

$= \Sigma\alpha_1\Sigma\frac{1}{\alpha_1} - n = \frac{p_1p_{n-1}}{p_n} - n$.

Theorem 5.5.1. Newton's theorem.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$$

and let $s_r = \alpha_1{}^r + \alpha_2{}^r + \cdots + \alpha_n{}^r$, where r is an integer ≥ 0. Then

(i) $s_r + p_1s_{r-1} + p_2s_{r-2} + \cdots + p_{r-1}s_1 + rp_r = 0$, if $1 \leq r < n$

(ii) $s_r + p_1s_{r-1} + p_2s_{r-2} + \cdots + p_ns_{r-n} = 0$, if $r \geq n$.

First we prove the following lemma.

Lemma. $f'(x) = nx^{n-1} + (s_1 + np_1)x^{n-2} + (s_2 + p_1s_1 + np_2)x^{n-3} + \ldots + (s_{n-1} + p_1s_{n-2} + \cdots + np_{n-1})$.

Proof. $f(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$.

$f'(x) = (x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n) + (x - \alpha_1)(x - \alpha_3)\ldots(x - \alpha_n) + \ldots + (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_{n-1})$

$= \frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \cdots + \frac{f(x)}{x-\alpha_n}$.

Let us use the method of synthetic division in order to express $\frac{f(x)}{x-\alpha}$ as a polynomial in x.

1	p_1	p_2	...	p_{n-1}	p_n
	α_1	$\alpha_1{}^2 + p_1\alpha_1$	...	$\alpha_1{}^{n-1} + \cdots + p_{n-2}\alpha_1$	...
1	$\alpha_1 + p_1$	$\alpha_1{}^2 + p_1\alpha_1 + p_2$	...	$\alpha_1{}^{n-1} + \cdots + p_{n-1}$	0

Since α_1 is a root, the remainder is zero.

Therefore $\frac{f(x)}{x-\alpha_1} = x^{n-1} + (\alpha_1 + p_1)x^{n-2} + (\alpha_1{}^2 + p_1\alpha_1 + p_2)x^{n-3} + \ldots + (\alpha_1^{n-1} + p_1\alpha_1{}^{n-2} + \cdots + p_{n-1})$.

We obtain similar expressions for $\frac{f(x)}{x-\alpha_2}, \frac{f(x)}{x-\alpha_3}, \ldots, \frac{f(x)}{x-\alpha_n}$.

Adding all these, we have $f'(x) = nx^{n-1} + (s_1 + np_1)x^{n-2} + (p_1s_1 + np_2)x^{n-3} + \ldots + (s_{n-1} + p_1s_{n-2} + \cdots + np_{n-1})$. $+ (s_2 +$

Proof of the theorem.

(i) We have $f'(x) = nx^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \cdots + p_{n-1}$.
Comparing this with the expression for $f'(x)$ obtained in the lemma, we have

$$\begin{aligned} s_1 + np_1 &= (n-1)p_1 \\ s_2 + p_1s_1 + np_2 &= (n-2)p_2 \\ s_3 + p_1s_2 + p_2s_1 + np_3 &= (n-3)p_3 \\ \cdots \quad &\cdots \quad \cdots \\ s_{n-1} + p_1s_{n-2} + p_2s_{n-3} + \cdots + np_{n-1} &= p_{n-1}. \end{aligned}$$

$$\begin{aligned} \text{Therefore } s_1 + p_1 &= 0 \\ s_2 + p_1s_1 + 2p_2 &= 0 \\ s_3 + p_1s_2 + p_2s_1 + 3p_3 &= 0 \\ \cdots \quad &\cdots \quad \cdots \\ s_{n-1} + p_1s_{n-2} + p_2s_{n-3} + \cdots + (n-1)p_{n-1} &= 0. \end{aligned}$$

All these together can be expressed as

$$s_r + p_1s_{r-1} + p_2s_{r-2} + \cdots + rp_r = 0, 1 \le r < n.$$

(ii) When $r = n$, putting $x = \alpha_1, \alpha_2 \ldots, \alpha_n$ successively in the equation $f(x) = 0$ and adding, we have

$$s_n + p_1s_{n-1} + p_2s_{n-2} + \cdots + np_n = 0. \quad \ldots \quad \text{(A)}$$

When $r > n$, let us consider the equation $x^{r-n}f(x) = 0$, i.e.,

$$x^r + p_1x^{r-1} + p_2x^{r-2} + \cdots + p_nx^{r-n} = 0.$$

This is an equation of degree r whose roots are $\alpha_1, \alpha_2, \ldots, \alpha_n$ and 0 (of multiplicity $r - n$).

Putting $x = \alpha_1, \alpha_2, \ldots, \alpha_n$ in succession and adding, we have

$$s_r + p_1s_{r-1} + p_2s_{r-2} + \cdots + p_ns_{r-n} = 0, \; r > n. \quad \ldots \quad \text{(B)}$$

Combining (A) and (B),

$$s_r + p_1s_{r-1} + p_2s_{r-2} + \cdots + p_ns_{r-n} = 0, \; r \ge n.$$

Note 1. $s_1, s_2, s_3, \ldots$ can be successively calculated in terms of the coefficients of the equation.

We have $s_1 + p_1 = 0$ and therefore $s_1 = -p_1$,
$s_2 + p_1s_1 + 2p_2 = 0$ and therefore $s_2 = p_1^2 - 2p_2$
and so on.

2. If none of $\alpha_1, \alpha_2, \ldots, \alpha_n$ be zero then $s_{-1}, s_{-2}, s_{-3}, \ldots$ can be calculated successively.

Let us consider the equation $x^{-1}f(x) = 0$, i.e.,

$$x^{n-1} + p_1x^{n-2} + \cdots + p_{n-1} + p_nx^{-1} = 0.$$

Putting $x = \alpha_1, \alpha_2, \ldots, \alpha_n$ in succession and adding, we have

$$s_{n-1} + p_1s_{n-2} + \cdots + p_{n-1}n + p_ns_{-1} = 0.$$

But $s_{n-1} + p_1s_{n-2} + \cdots + (n-1)p_{n-1} = 0$.
Therefore $p_ns_{-1} + p_{n-1} = 0$. This gives s_{-1}.

Let us consider the equation $x^{-2}f(x) - 0$, i.e.,

$$x^{n-2} + p_1x^{n-3} + \cdots + p_{n-2} + p_{n-1}x^{-1} + p_nx^{-2} = 0.$$

Putting $x = \alpha_1, \alpha_2, \ldots, \alpha_n$ in succession and adding, we have

$$s_{n-2} + p_1s_{n-3} + \cdots + np_{n-2} + p_{n-1}s_{-1} + p_ns_{-2} = 0.$$

But $s_{n-2} + p_1s_{n-3} + \cdots + (n-2)p_{n-2} = 0$.
Therefore $p_ns_{-2} + p_{n-1}s_{-1} + 2p_{n-2} = 0$. This gives s_{-2}.

Worked Examples (continued).

5. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of the equation $x^4 + p_2x^2 + p_3x + p_4 = 0$, find the value of

(i) $\Sigma\alpha^3$, (ii) $\Sigma\alpha^4$, (iii) $\Sigma\alpha^6$.

(i) By Newton's theorem, $s_3 + p_2s_1 + 3p_3 = 0$.
Here $s_1 = 0$. Therefore $s_3 = -3p_3$.

(ii) By Newton's theorem, $s_4 + p_2s_2 + p_3s_1 + 4p_4 = 0$.
Here $s_1 = 0$ and $s_2 = -2p_2$. Therefore $s_4 = 2(p_2^2 - 2p_4)$.

(iii) By Newton's theorem, $s_6 + p_2s_4 + p_3s_3 + p_4s_2 = 0$

$$\begin{aligned} \text{or, } s_6 &= -p_2s_4 - p_3s_3 - p_4s_2 \\ &= p_2(4p_4 - 2p_2^2) + 3p_3^2 + 2p_4p_2 \\ &= 6p_4p_2 + 3p_3^2 - 2p_2^3. \end{aligned}$$

6. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of the equation $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$, find the value of

(i) $\Sigma\alpha_1^2\alpha_2$, (ii) $\Sigma\alpha_1^2\alpha_2^2$, (iii) $\Sigma\alpha_1^3\alpha_2$.

Let $s_m = \alpha_1^m + \alpha_2^m + \alpha_3^m + \alpha_4^m$.

(i) $$\begin{aligned} s_2s_1 &= (\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2)(a_1 + a_2 + a_3 + a_4) \\ &= \Sigma\alpha_1^3 + \Sigma\alpha_1^2a_2 = s_3 + \Sigma\alpha_1^2a_2. \end{aligned}$$

Therefore $\Sigma\alpha_1^2a_2 = s_1s_2 - s_3$.

(ii) $$\begin{aligned} s_2s_2 &= (\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2)^2 \\ &= \Sigma\alpha_1^4 + 2\Sigma\alpha_1^2\alpha_2^2 = s_4 + 2\Sigma\alpha_1^2a_2. \end{aligned}$$

Therefore $\Sigma\alpha_1^2a_2^2 = \frac{1}{2}(s_2^2 - s_4)$.

(iii) $s_3 s_1 = (\alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_4^3)(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$
$= \Sigma\alpha_1^4 + \Sigma\alpha_1^3\alpha_2 = s_4 + \Sigma\alpha_1^3\alpha_2.$

Therefore $\Sigma\alpha_1^3\alpha_2 = s_3 s_1 - s_4$.

Note. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation
$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0$,
$\Sigma\alpha_1^m\alpha_2^p$ can be calculated when m and p are positive integers.

When $m \neq p$, $\Sigma\alpha_1^m \Sigma\alpha_1^p = \Sigma\alpha_1^{m+p} + \Sigma\alpha_1^m\alpha_2^p$
or, $\Sigma\alpha_1^m\alpha_2^p = s_m s_p - s_{m+p}$.

When $m = p$, $\Sigma\alpha_1^m)^2 = \Sigma\alpha_1^{2m} + 2\Sigma\alpha_1^m\alpha_2^m$
or, $2\Sigma\alpha_1^m\alpha_2^m = \frac{1}{2}(s_m^2 - s_{2m})$.

Exercises 5C

1. Solve the equations

(i) $x^3 + 6x^2 - 3x - 18 = 0$,

(ii) $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$,

(iii) $2x^4 + 8x^3 + 3x^2 + 4x + 1 = 0$,

given that the sum of two of the roots is zero.

2. Solve the equations

(i) $x^3 + 5x^2 + 7x + 2 = 0$,

(ii) $x^4 + 2x^3 + 5x^2 + 4x + 3 = 0$,

(iii) $2x^4 + 2x^3 - 33x^2 - 10x + 5 = 0$,

given that the product of two of the roots is 1.

3. Solve the equations

(i) $x^3 + 6x^2 + 11x + 6 = 0$,

(ii) $4x^4 - 4x^3 - 21x^2 + 11x + 10 = 0$,

(iii) $4x^4 + 20x^3 + 35x^2 + 25x + 6 = 0$,

given that the roots are in arithmetic progression.

4. Solve the equations

(i) $3x^3 - 26x^2 + 52x - 24 = 0$,

(ii) $x^4 - 5x^3 - 30x^2 + 40x + 64 = 0$,

(iii) $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$,

(iv) $3x^4 + 20x^3 - 70x^2 - 60x + 27 = 0$,

given that the roots are in geometric progression.

5. Solve the equations

(i) $3x^4 + 5x^3 + 8x^2 - x + 3 = 0$,

(ii) $x^4 + 10x^3 + 26x^2 + 6x - 3 = 0$,

given that the product of two of the roots is 3.

6. Solve the equations

(i) $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$,

(ii) $x^4 - 8x^3 + 21x^2 - 20x + 6 = 0$,

given that the sum of two of the roots is equal to the sum of the other two.

7. Solve the equations

(i) $x^4 + 3x^3 - 4x^2 - 9x + 9 = 0$,

(ii) $2x^4 + x^3 + 2x^2 + 3x + 18 = 0$,

given that the product of two of the roots is equal to the product of the other two.

8. Solve the equations

(i) $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$,

(ii) $x^4 + 2x^3 - 18x^2 + 6x + 9 = 0$,

(iii) $2x^4 + 3x^3 - 19x^2 + 6x + 8 = 0$,

given that the ratio of two of the roots is equal to the ratio of the other two.

9. Determine k and solve the equation if the roots are in arithmetic progression.

(i) $8x^3 - 12x^2 - kx + 3 = 0$

(ii) $x^4 - 8x^3 + kx^2 + 8x - 15 = 0$.

10. Find the relation among the coefficients of the equation
$$ax^3 + 3bx^2 + 3cx + d = 0$$
if its roots be in (i) arithmetic progression, (ii) geometric progression, (iii) harmonic progression.

11. Find the relation among the coefficients of the equation
$$x^4 + px^3 + qx^2 + rx + s = 0$$
if its roots $\alpha, \beta, \gamma, \delta$ be connected by the relation

(i) $\alpha + \beta = 0$, (ii) $\alpha + \beta = \gamma + \delta$, (iii) $\alpha\beta + \gamma\delta = 0$,
(iv) $\alpha\beta + 1 = 0$, (v) $\alpha\beta = \gamma\delta$.

12. If the equation $x^3 + px^2 + qx + r = 0$ has a root $\alpha + i\alpha$ where p, q, r and α are real, prove that $(p^2 - 2q)(q^2 - 2pr) = r^2$.

Hence solve the equations (i) $x^3 - x^2 - 4x + 24 = 0$,

(ii) $x^3 - 7x^2 + 20x - 24 = 0$.

13. If the equation $x^4+px^3+qx^2+rx+s=0$ has roots of the form $\alpha\pm i\alpha, \beta\pm i\beta$ where α, β are real, prove that $p^2-2q=0$ and $r^2-2qs=0$.

Hence solve the equations

(i) $x^4+4x^3+8x^2-24x+36=0$,

(ii) $x^4+6x^3+18x^2+24x+16=0$.

14. Solve the equations

(i) $x^4+2x^3+5x^2+4x+4=0$,

(ii) $4x^4+20x^3+13x^2-30x+9=0$,

given that each has two distinct pairs of equal roots.

15. (a) If the product of two roots of the equation $x^4+px^3+qx^2+rx+s=0$ be equal to the product of the other two, prove that $r^2=p^2s$.

If $p\neq 0$ show that the equation can be solved by the substitution

$$x+\frac{r}{px}=t.$$

[**Hint.** (i) The equation is $(x^2+\frac{r^2}{p^2x^2})+p(x+\frac{r}{px})+q=0$.]

(b) Solve the equation $x^4-12x^3+47x^2-72x+36=0$, given that the product of two of the roots is equal to the product of the other two.

Note. If the roots $\alpha, \beta, \gamma, \delta$ of the equation $x^4+px^3+qx^2+rx+s=0$ be known to be in geometric progression then $\alpha\delta=\beta\gamma$. In this case the condition $r^2=p^2s$ is necessarily satisfied and therefore the equation can be solved by the substituion $x+\frac{r}{px}=t$.

(c) Solve the equations

(i) $x^4-5x^3-30x^2+40x+64=0$,

(ii) $3x^4+20x^3-70x^2-60x+27=0$,

given that the roots are in geometric progression.

[**Hint.** (ii) The equation is $3(x^2+\frac{9}{x^2})+20(x-\frac{3}{x})-70=0$.]

16. If α, β, γ be the roots of the equation $x^3+px^2+qx+r=0$, find the value of

(i) $\Sigma\alpha^2\beta^2$, (ii) $\Sigma\alpha^3\beta^3$, (iii) $\Sigma(\beta+\gamma-\alpha)^3$,
(iv) $\Sigma\alpha^4$, (v) $\Sigma\alpha^2\beta^2\gamma$, (vi) $\Sigma(\alpha-\beta)^2\gamma$,
(vii) $\Sigma(\alpha-\beta)^2\gamma^2$, (viii) $(\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)(\alpha+\beta-2\gamma)$.

17. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4+px^3+qx^2+rx+s=0$, find the value of

(i) $\Sigma\alpha^3\beta\gamma$, (ii) $\Sigma\alpha^2\beta^2\gamma^2$, (iii) $\Sigma(\alpha-\beta)^2\gamma\delta$,
(iv) $\Sigma(\alpha-\beta)^2\gamma^2\delta^2$, (v) $(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)$.

18. If α, β, γ be the roots of the equation $x^3+qx+r=0 \quad (r\neq 0)$, show that

(i) $\Sigma\frac{\alpha^2}{\beta\gamma}=3$, (ii) $\Sigma\frac{1}{(\alpha+\beta)^2}=\frac{q^2}{r^2}$,

(iii) $\Sigma\frac{\alpha^2}{\beta}=\frac{2q^2}{r}$, (iv) $\Sigma\frac{2\beta\gamma-\alpha^2}{\beta+\gamma-\alpha}=\frac{q^2}{r}$.

19. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4+px^3+qx^2+rx+s=0 (s\neq 0)$, find the value of

(i) $\Sigma\frac{\alpha}{\beta}$, (ii) $\Sigma\frac{\alpha\beta}{\gamma}$, (iii) $\Sigma\frac{\alpha\beta}{\gamma\delta}$, (iv) $\Sigma\frac{\alpha^2}{\beta^2}$.

20. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4+px^3+qx^2+rx+s=0$, prove that

(i) $(\alpha\beta+\gamma\delta)(\beta\gamma+\alpha\delta)+(\beta\gamma+\alpha\delta)(\gamma\alpha+\beta\delta)+(\gamma\alpha+\beta\delta)(\alpha\beta+\gamma\delta)=pr-4s$,

(ii) $(\alpha\beta+\gamma\delta)(\beta\gamma+\alpha\delta)(\gamma\alpha+\beta\delta)=r^2-4qs+p^2s$.

Show that the equation whose roots are $\alpha\beta+\gamma\delta, \beta\gamma+\alpha\delta, \gamma\alpha+\beta\delta$ is $x^3-qx^2+(pr-4s)x-(r^2-4qs+p^2s)=0$.

Deduce that

(i) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)(\alpha+\delta)(\beta+\delta)(\gamma+\delta)=pqr-r^2-p^2s$,

(ii) $(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2=2q^2-6pr+24s$.

21. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n=0$, show that

(i) $\Sigma{\alpha_1}^2\alpha_2=-p_1p_2+3p_3$, (ii) $\Sigma{\alpha_1}^2\alpha_2\alpha_3=-p_1p_3-4p_4$,

(iii) $\Sigma\frac{{\alpha_1}^2}{\alpha_2}=\frac{p_1p_n-({p_1}^2-2p_2)p_{n-1}}{p_n}$,

(iv) $\Sigma\frac{{\alpha_1}^2}{\alpha_2}=\frac{p_np_{n-1}-p_1({p_{n-1}}^2-2p_np_{n-2})}{{p_n}^2}$.

22. If $\alpha+\beta+\gamma=0$, prove that

(i) $\frac{\alpha^5+\beta^5+\gamma^5}{5}=\frac{\alpha^3+\beta^3+\gamma^3}{3}\cdot\frac{\alpha^2+\beta^2+\gamma^2}{2}$,

(ii) $\frac{\alpha^7+\beta^7+\gamma^7}{7}=\frac{\alpha^5+\beta^5+\gamma^5}{5}\cdot\frac{\alpha^2+\beta^2+\gamma^2}{2}$.

23. If $\alpha+\beta+\gamma+\delta=0$, prove that

$$\frac{\alpha^5+\beta^5+\gamma^5+\delta^5}{5}=\frac{\alpha^3+\beta^3+\gamma^3+\delta^3}{3}\cdot\frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{2}.$$

24. (i) If $\alpha+\beta+\gamma=1, \alpha^2+\beta^2+\gamma^2=3$ and $\alpha^3+\beta^3+\gamma^3=7$, prove that $\alpha^4+\beta^4+\gamma^4=11$.

(ii) If $a+b+c+d=1, a^2+b^2+c^2+d^2=3, a^3+b^3+c^3+d^3=7$ and $a^4+b^4+c^4+d^4=15$, prove that $a^5+b^5+c^5+d^5=26$.

25. If s_m denote the sum of mth powers of the roots of the equation $x^3+qx+r=0$, prove that

(i) $s_3=-3r$, (ii) $s_4=2q^2$, (iii) $s_5=5qr$, (iv) $s_7=-7q^3r$.

26. If s_m denote the sum of mth powers of the roots of the equation $x^4+qx^2+r=0$, prove that

(i) $s_4=2q^2-4s$, (ii) $s_6=6qs-2q^3$.

5.6. Transformation of equations.

When an equation is given it is possible, without knowing its individual roots, to obtain a new equation whose roots are connected with those of the equation by some assigned relation. The method of finding the new equation is said to be a *transformation*. Such a transformation sometimes helps us to study the nature of the roots of the given equation which would have been otherwise a difficult job.

We will discuss some typical transformation and then take up the general problem.

5.6.1. To transform an equation whose roots are $\alpha_1, \alpha_2, \ldots, \alpha_n$ into another whose roots are $m\alpha_1, m\alpha_2, \ldots, m\alpha_n$.

Let the equation whose roots are $\alpha_1, \alpha_2, \ldots . \alpha_n$ be

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0.$$

Let $y = m\alpha_1$. Then $\alpha_1 = y/m$.

Since α_1 is a root of the given equation,

$$\alpha_1{}^n + p_1 \alpha_1{}^{n-1} + p_2 \alpha_1{}^{n-2} + \cdots + p_n = 0.$$

Therefore $(\frac{y}{m})^n + p_1(\frac{y}{m})^{n-1} + p_2(\frac{y}{m})^{n-2} + \cdots + p_n = 0$

or, $y^n + p_1 m y^{n-1} + p_2 m^2 y^{n-2} + \cdots + p_n m^n = 0.$ ... (A)

This is the transformed equation.

[**Explanation.** $m\alpha_1$ is obviously a root of the equation (A). We would obtain the same equation (A) if we put $y = m\alpha_i$, where α_i is any root of the given equation. This proves that $m\alpha_1, m\alpha_2, \ldots, m\alpha_n$ are the roots of the equation (A). Since the equation (A) is of degree n, the roots $m\alpha_1, m\alpha_2, \ldots, m\alpha_n$ are the only roots of it.]

Note 1. We observe that the successive coefficients of the transformed equation are obtained by multiplying the successive coefficients of the given equation, beginning from the 1st, by $1, m, m^2, \ldots, m^n$ respectively.

Corollary. The transformed equation whose roots are $-\alpha_1, -\alpha_2, \ldots -\alpha_n$ is obtained by changing the signs of the coefficients beginning from the second, alternately.

Note 2. The transformation is useful for the purpose of removing fractional coefficients of an equation or reducing the leading coefficient of an equation to unity. We are to choose a suitable multiplier of the roots so that the transformed equation will have all integral coefficients with the leading coefficient unity.

Worked examples.

1. Multiply the roots of the equation $x^4 + \frac{1}{2}x^3 - \frac{1}{3}x^2 + \frac{1}{4}x + \frac{5}{12} = 0$ by a suitable constant so that the fractional coefficients of the equation may be removed.

Multiplying the roots by 6 the transformed equation becomes

$$x^4 + 6.\tfrac{1}{2}x^3 - 36.\tfrac{1}{3}x^2 + 216.\tfrac{1}{4}x + 1296.\tfrac{5}{12} = 0$$

or, $x^4 + 3x^3 - 12x^2 + 54x + 540 = 0.$

2. One root of the equation $32x^3 - 14x + 3 = 0$ is double another. Solve the equation.

The equation whose roots are double those of the given equation is $32x^3 - 56x + 24 = 0$ or, $4x^3 - 7x + 3 = 0$.

Hence the equations $32x^3 - 14x + 3 = 0$ and $4x^3 - 7x + 3 = 0$ have a common root.

The h.c.f. of $32x^3 - 14x + 3$ and $4x^3 - 7x + 3$ is $2x - 1$.

Therefore $\frac{1}{2}$ and $\frac{1}{4}$ are roots of the given equation. The other root is $-\frac{3}{4}$. Hence the roots of the given equation are $\frac{1}{2}, \frac{1}{4}, -\frac{3}{4}$.

5.6.2. To transform an equation into one whose roots are reciprocal of the roots of a given equation.

Let the given equation be $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ and let the roots be $\alpha_1, \alpha_2, \ldots, \alpha_n$.

It is assumed that none of the roots is zero.

Let $y = (1/\alpha_1)$. Then $\alpha_1 = (1/y)$.

Since α_1 is a root, $a_0\alpha_1{}^n + a_1\alpha_1{}^{n-1} + a_2\alpha_1{}^{n-2} + \cdots + a_{n-1}\alpha_1 + a_n = 0.$

Therefore $a_0(\frac{1}{y})^n + a_1(\frac{1}{y})^{n-1} + a_2(\frac{1}{y})^{n-2} + \cdots + a_{n-1}(\frac{1}{y}) + a_n = 0$

or, $a_n y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0 = 0.$

This is the transformed equation.

Note. The coefficients of the given equation appear in the reverse order in the transformed equation.

Worked Examples (continued).

3. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $x_n + p_1x^{n-1}p_2x^{n-2} + \cdots + p_n = 0$ $(p_n \neq 0)$, find the value of $\Sigma(1/\alpha_1{}^2)$.

Let $\alpha_i' = (1/\alpha_i), i = 1, 2, \ldots, n.$

$\alpha_1', \alpha_2', \ldots, \alpha_n'$ are the roots of the equation

$$p_n y^n + p_{n-1}y^{n-1} + \cdots + p_1 y + 1 = 0.$$

Therefore $\Sigma\alpha_1' = -\frac{p_{n-1}}{p_n}$ and $\Sigma{\alpha_1'}^2 = (\Sigma\alpha_1')^2 - 2\Sigma\alpha_1'\alpha_2'$

$= \left(\frac{p_{n-1}}{p_n}\right)^2 - 2\frac{p_{n-2}}{p_n} = \frac{p_{n-1}^2 - 2p_n p_{n-2}}{p_n^2}$.

Hence $\Sigma(1/{\alpha_1}^2) = \frac{p_{n-1}^2 - 2p_n p_{n-2}}{{p_n}^2}$.

5.6.3. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$f(x) \equiv a_0 x_n + a_1 x^n + x_1 x^{n-1} + \cdots + a_n = 0.$$

To find the equation whose roots are $\alpha_1 - h, \alpha_2 - h, \ldots, \alpha_n - h$, where h is a constant.

Let $y = \alpha_1 - h$. Then $\alpha_1 = y + h$.

Since α_1 is a root, ${a_0\alpha_1}^n + a_1{\alpha_1}^{n-1} + \cdots + a_n = 0$.

or, $a_0(y+h)^n + a_1(y+h)^{n-1} + \cdots + a_n = 0$.

This can be expressed as $A_0 x_n + A_1 x^{n-1} + \cdots + A_n = 0$, where A_n is the remainder when $f(x)$ is divided by $x - h$ and let $q_1(x)$ be the quotient; A_{n-1} is the remainder when the quotient $q_1(x)$ is divided by $x - h$ and let $q_2(x)$ be the quotient.

Repeating the same process, $A_{n-2}, A_{n-3}, \ldots, A_1$ are obtained as the successive remainders and finally $A_0 = a_0$.

Note. Here the transformation that is applied to the equation is given by $x = y + h$.

Worked Examples (continued).

4. Find the equation whose roots are the roots of the equation

$$x^3 + 3x^2 - 8x + 1 = 0,$$

(i) each diminished by 4, (ii) each increased by 1.

(i) Let α, β, γ be the roots of the given equation. To find the equation whose roots are $\alpha - 4, \beta - 4, \gamma - 4$.

Let $y = \alpha - 4$. Then $\alpha = y + 4$.

Since α is a root of the given equation, $\alpha^3 + 3\alpha^2 - 8\alpha + 1 = 0$.

Therefore $(y+4)^3 + 3(y+4)^2 - 8(y+4) + 1 = 0$

or, $y^3 + 15y^2 + 64y + 81 = 0$. This is the required equation.

(ii) To find the equation whose roots are $\alpha + 1, \beta + 1, \gamma + 1$.

Let $y = \alpha + 1$. Then $\alpha = y - 1$.

Since α is a root of the given equation, $\alpha^3 + 3\alpha^2 - 8\alpha + 1 = 0$.

Therefore $(y-1)^3 + 3(y-1)^2 - 8(y-1) + 1 = 0$

or, $y^3 - 11y + 11 = 0$.

This is the required equation.

5. Find the equation whose roots are the roots of the equation $x^4 - 8x^2 + 8x + 6 = 0$, each diminished by 2.

Let $\alpha, \beta, \gamma, \delta$ be the roots of the given equation. To find the equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2, \delta - 2$.

Let $y = \alpha - 2$. Then $\alpha = y + 2$.

Since α is a root of the given equation, $\alpha^4 - 8\alpha^2 + 8\alpha + 6 = 0$.

Therefore $(y+2)^4 - 8(y+2)^2 + 8(y+2) + 6 = 0$.

This can be expressed as $A_0 y^4 + A_1 y^3 + A_2 y^2 + A_3 y + A_4 = 0$.

The calculation of $A_0, A_1, \ldots, A_4$ is shown in the following table:

2	1	0	-8	8	6	
		2	4	-8	0	
	1	2	-4	0	6	$= A_4$
		2	8	8		
	1	4	4	8		$= A_3$
		2	12			
	1	6	16			$= A_2$
		2				
	1	8	$= A_1$			
	1	$= A_0$				

Here the required equation is $x^4 + 8y^3 + 16y^2 + 8y + 6 = 0$.

Note. The transformation can be utilised to remove a specified term from an equation. We are to choose a suitable h so that the ith coefficient of the transformed equation may be zero. Such a transformation often facilitates the solution of the given equation.

6. Transform the equation $x^4 + 4x^3 + 7x^2 + 6x - 4 = 0$ into one which shall want the second term and hence solve the given equation.

Let us apply the transformation $x = y + h$ so that the transformed equation may want the second term.

The transformed equation is

$(y+h)^4 + 4(y+h)^3 + 7(y+h)^2 + 6(y+h) - 4 = 0$

or, $y^4 + y^3(4h+4) + y^2(6h^2 + 12h + 7) + y(4h^3 + 12h^2 + 14h + 6) + (h^4 + 4h^3 + 7h^2 + 6h - 4) = 0$.

By the given condition $4h + 4 = 0$, i.e., $h = -1$.

The equation reduces to $y^4 + y^2 - 6 = 0$.

The roots of the transformed equation are $\pm\sqrt{2}, \pm\sqrt{3}i$.

Hence the roots of the given equation are $-1 \pm \sqrt{2}, -1 \pm \sqrt{3}i$.

5.6.4. Transformation in general.

Given an equation $f(x) = 0$ we are to obtain an equation $\phi(y) = 0$ whose roots are connected with the roots of the given equation by a relation $\psi(x, y) = 0$.
$\phi(y) = 0$ is obtained by eliminating x between $f(x) = 0$ and $\psi(x, y) = 0$.

The method is illustrated in the following examples.

Worked Examples (continued).

7. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are $\alpha\beta + \beta\gamma, \beta\gamma + \gamma\alpha, \gamma\alpha + \alpha\beta$.

Let $y = \alpha\beta + \beta\gamma$. Then $y = q - \gamma\alpha = q + \frac{r}{\beta}$.
Therefore $\beta = \frac{r}{y-q}$ [Note that here $\phi(x, y) = 0$ is $x(y - q) - r = 0$.]
Since β is a root of the given equation, $\beta^3 + p\beta^2 + q\beta + r = 0$.
Therefore $(\frac{r}{y-q})^3 + p(\frac{r}{y-q})^2 + q(\frac{r}{y-q}) + r = 0$
or, $(y - q)^3 + q(y - q)^2 + pr(y - q) + r^2 = 0$.
This is the required equation.

8. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are

(i) $\beta^2 + \gamma^2 - \alpha^2, \gamma^2 + \alpha^2 - \beta^2, \alpha^2 + \beta^2 - \gamma^2$

(ii) $\frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{1}{\alpha^2}, \frac{1}{\gamma^2} + \frac{1}{\alpha^2} - \frac{1}{\beta^2}, \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{1}{\gamma^2}$.

(i) Let $y = \beta^2 + \gamma^2 - \alpha^2$. Then $y = (\beta^2 + \gamma^2 + \alpha^2) - 2\alpha^2$
$= (p^2 - 2q) - 2\alpha^2$

or, $\alpha^2 = \frac{p^2 - 2q - y}{2}$.

Since α is a root of the given equation, $\alpha^3 + p\alpha^2 + q\alpha + r = 0$.

Therefore $\alpha^2(\alpha^2 + q)^2 - (p\alpha^2 + r)^2 = 0$
or, $(p^2 - 2q - y)(p^2 - y)^2 - 2\{p(p^2 - 2q - y) + 2r\}^2 = 0$. ... (A)

This is the required equation.

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are the roots of the equation
$rx^3 + qx^2 + px + 1 = 0$
or, $x^3 + \frac{q}{r}x^2 + \frac{p}{r}x + \frac{1}{r} = 0$.

Hence the required equation is obtained from (A) if we replace p, q, r by $\frac{q}{r}, \frac{p}{r}, \frac{1}{r}$ respectively.

9. If α, β, γ be the roots of the equation $x^3 + qx + r = 0$ $(r \neq 0)$, find the equation whose roots are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}, \frac{\beta}{\gamma}, \frac{\gamma}{\beta}, \frac{\gamma}{\alpha}, \frac{\alpha}{\gamma}$.

Let $y = \frac{\alpha}{\beta}$. Then $\alpha = \beta y$.
Since α is a root of the given equation , $\alpha^3 + q\alpha + r = 0$.
Therefore $\beta^3 y^3 + q\beta y + r = 0$ ... (i)
Since β is a root, $\beta^3 + q\beta + r = 0$ ... (ii)
From (i) and (ii) $\frac{\beta^3}{qry - qr} = \frac{\beta}{r(1-y^3)} = \frac{1}{y^3 q - yq}$.
Hence $\beta = \frac{r(1-y^3)}{qy(y^2-1)} = -\frac{r(y^2+y+1)}{qy(y+1)}$, $\beta^3 = \frac{qr(y-1)}{qy(y^2-1)} = \frac{r}{y(y+1)}$.
Therefore $r^2(y^2 + y + 1)^3 + q^3 y^2 (y + 1)^2 = 0$.
This is the required equation.

10. Solve the equation $x^4 - 4x^3 - 4x^2 - 4x - 5 = 0$, given that two roots α, β are connected by the relation $2\alpha + \beta = 3$.

The relation is $2\alpha = 3 - \beta$.
Let us find the equation whose roots are $2\alpha, 2\beta, 2\gamma, 2\delta$ and the equation whose roots are $3 - \alpha, 3 - \beta, 3 - \gamma, 3 - \delta$.

The equation whose roots are $2\alpha, 2\beta, 2\gamma, 2\delta$ is
$y^4 - 8y^3 - 16x^2 - 32y - 80 = 0$, say $f(y) = 0$ ... (i)

The equation whose roots are $3 - \alpha, 3 - \beta, 3 - \gamma, 3 - \delta$ is
$(3 - 4)^4 - 4(3 - y)^3 - 4(3 - y)^2 - 4(3 - y) - 5 = 0$
or, $y^4 - 8y^3 + 14y^2 + 28y - 80 = 0$, say $\phi(y) = 0$ ... (ii)

The h.c.f. of the polynomials $f(y)$ and $\phi(y)$ is $y + 2$.
Therefore -2 is a common root of the equations (i) and (ii).
Consequently, $2\alpha = 3 - \beta = -2$, i.e.,$\alpha = -1, \beta = 5$.

Therefore $(x + 1)(x - 5)$ is a factor of $x^4 - 4x^3 - 4x^2 - 4x - 5$.
$x^4 - 4x^3 - 4x^2 - 4x - 5 = (x + 1)(x - 5)(x^2 + 1)$.
$x^2 + 1 = 0$ gives $x = \pm i$.
Hence the roots of the equation are $-1, 5, \pm i$.

Exercises 5D

1. Multiply the roots of the equation by a suitable constant so that the fractional coefficients of the equation may be removed

(i) $x^3 - 2x = \frac{3}{40} = 0$, (ii) $x^3 + \frac{1}{2}x^2 + \frac{5}{36}x + \frac{7}{72} = 0$.

2. If α, β, γ be the roots of the equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0 \quad (a_3 \neq 0),$$

find the value of (i) $\Sigma 1/\alpha^2$, (ii) $\Sigma 1/\alpha^2\beta^2$, (iii) $\Sigma 1/\alpha^3$.

[**Hint.** The roots of the equation $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.]

3. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, prove that

(i) $\Sigma\alpha^2\beta = 3r - pq$, (ii) $\Sigma\alpha^3\beta\gamma = ps + 2qr - p^2r$,

(iii) $\Sigma(\alpha - \beta)^2 = 3p^2 - 8q$.

Deduce from (i),(ii) and (iii) respectively that

(iv) $\Sigma\alpha^2\beta^2\gamma = 3ps - qr$, (v) $\Sigma\alpha^3\beta^2\gamma^2 = rs + 2pqs - pr^2$,

(vi) $\Sigma(\alpha - \beta)^2\gamma^2\delta^2 = 3r^2 - 8qs$.

4. $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0$. Prove that

(i) $\Sigma(\alpha_1 - \alpha_2)^2\alpha_3\alpha_4 \ldots \alpha_n = (-1)^n(p_1p_{n-1} - n^2p_n)$,

(ii) $\Sigma(\alpha_1 - \alpha_2)^2\alpha_3^2\alpha_4^2 \ldots \alpha_n^2 = (n-1)p^2_{n-1} - 2np_np_{n-2}$.

[**Hint.** (ii) $\Sigma(\alpha_1 - \alpha_2)^2\alpha_3^2\alpha_4^2 \ldots \alpha_n^2 = p_n^2[\Sigma(\frac{1}{\alpha_1} - \frac{1}{\alpha_2})^2] = p_n^2[(n-1)\Sigma\frac{1}{\alpha_1^2} - 2\Sigma\frac{1}{\alpha_1\alpha_2}]$.]

5. If α, β, γ are the roots of the equation $2x^3 + 3x^2 + x + 1 = 0$, find the value of (i) $\Sigma\frac{1}{\alpha^2+3\alpha+2}$, (ii) $\Sigma\frac{\alpha}{\alpha^2+3\alpha+2}$, (iii) $\Sigma\frac{\alpha^2}{\alpha^2+3\alpha+2}$.

[**Hint.** (iii) $\Sigma\frac{\alpha^2}{\alpha^2+3\alpha+2} = 3 + \Sigma\frac{1}{\alpha+1} - 4\Sigma\frac{1}{\alpha+2}$.]

6. The roots of the equation $x^3 + px^2 + qx + r = 0 \quad (r \neq 0)$ are α, β, γ. Find the equation whose roots are

(i) $1/\alpha + 1/\beta - 1/\gamma, 1/\beta + 1/\gamma - 1/\alpha, 1/\gamma + 1/\alpha - 1/\beta$,

(ii) $\alpha\beta + 1/\gamma, \beta\gamma + 1/\alpha, \gamma\alpha + 1/\beta$,

(iii) $\alpha - \frac{\beta\gamma}{\alpha}, \beta - \frac{\gamma\alpha}{\beta}, \gamma - \frac{\alpha\beta}{\gamma}$,

(iv) $\frac{\alpha+\beta}{\gamma}, \frac{\beta+\gamma}{\alpha}, \frac{\gamma+\alpha}{\beta}$.

7. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, find the equation whose roots are

(i) $\alpha(\beta+\gamma), \beta(\gamma+\alpha), \gamma(\alpha+\beta)$;

(ii) $(\alpha-\beta)(\alpha-\gamma), (\beta-\alpha)(\beta-\gamma), (\gamma-\alpha)(\gamma-\beta)$;

(iii) $\alpha^2 + \beta^2, \beta^2 + \gamma^2, \gamma^2 + \alpha^2$;

(iv) $\beta + \gamma - 2\alpha, \gamma + \alpha - 2\beta, \alpha + \beta - 2\gamma$.

because symmetric function. then we take one root and find the value of b & p / 8.

8. If α, β, γ be the roots of the equation $x^3 + qx + r = 0 \quad (r \neq 0)$, find the equation whose roots are $\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}, \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

9. If α, β, γ be the roots of the equation $x^3 - 2x^2 + 3x - 1 = 0$, find the equation whose roots are $\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha}, \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta}, \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}$.

10. Solve the equations

(i) $16x^3 - 44x^2 + 36x - 9 = 0$,

(ii) $15x^4 - 16x^3 - 56x^2 + 64x - 16 = 0$,

(iii) $40x^4 - 22x^3 - 21x^2 + 2x + 1 = 0$,

given that the roots are in harmonic progression.

[**Hint.** The roots of $f(1/x) = 0$ are in A.P.]

11. Solve the equation $4x^4 + 4x^3 + 3x^2 - x - 1 = 0$, given that the difference between two of its roots is 1.

12. Solve the equations

(i) $3x^3 + 14x^2 + 17x + 6 = 0$,

(ii) $x^4 - 4x^3 + 2x^2 + x + 6 = 0$,

given that two roots α, β are connected by the relation $2\alpha = 3\beta$.

13. Solve the equations

(i) $2x^3 - 9x^2 + 7x + 6 = 0$,

(ii) $2x^4 - 3x^3 - 3x - 2 = 0$,

given that two roots α, β are connected by the relation $2\alpha + \beta = 1$.

14. Transform the equation $x^3 + 6x^2 + 12x + 35 = 0$ into one which shall want the second term and hence solve the given equation.

15. Transform the equation $x^4 + 4x^3 + 9x^2 + 10x - 6 = 0$ into one which shall want the second term and hence solve the given equation.

16. Find the relation among the coefficients of the equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

so that the second term and the third term may be removed by the same tranformation $x = y + h$.

17. Find the relation among the coefficients of the equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$$

so that the second term and the fourth term may be removed by the same tranformation $x = y + h$.

18. Obtain the equation whose roots are the roots of the equation

$$4x^3 - 8x^2 - 19x + 26 = 0$$

each diminished by 2. Use Descartes' rule of signs to both the equations to find the exact number of positive and negative roots of the given equation.

19. Obtain the equation whose roots exceed the roots of the equation

$$x^4 + 3x^2 + 8x + 3 = 0$$

by 1. Use Descartes' rule of signs to both the equations to find the exact number of real and complex roots of the given equation.

20. Find the equation whose roots are squares of the roots of the equation

$$x^4 - x^3 + 2x^2 - x + 1 = 0$$

and use Decartes' rule of signs to the resulting equation to deduce that the given equation has no real root.

21. If the equation whose roots are squares of the roots of the cubic

$$x^3 - ax^2 + bx - 1 = 0$$

is identical with this cube, prove that either $a = b = 0$, or $a = b = 3$, or a and b are the roots of the equation $t^2 + t + 2 = 0$.

22. Find the equation whose roots are cubes of the roots of the cubic

$$x^3 + 3x^2 + 2 = 0.$$

23. Show that the cubes of the roots of the cubic $x^3 + ax^2 + bx + ab = 0$ are the roots of the cubic $x^3 + a^3x^2 + b^3x + a^3b^3 = 0$.

24. The roots of the equation $x^3 + px^2 + qx + r = 0$ are α, β, γ. Find the equation whose roots are $\alpha\beta - \gamma^2, \beta\gamma - \alpha^2, \gamma\alpha - \beta^2$.

Deduce the condition that the roots of the given equation may be in geometric progression.

25. The roots of the equation $x^3 + px^2 + qx + r = 0$ are α, β, γ. Find the equation whose roots are $\alpha + \beta - 2\gamma, \beta + \gamma - 2\alpha, \gamma + \alpha - 2\beta$.

Deduce the condition that the roots of the given equation may be in arithmetic progression.

26. If α be a root of the equation $x^3 - 3x - 1 = 0$, prove that the other roots are $2 - \alpha^2, \alpha^2 - \alpha - 2$.

[**Hint.** The substitution $y = 2 - x^2$ transforms the equation into itself.]

27. If α be a root of the equation $x^3 + 3x^2 - 6x + 1 = 0$, prove that the other roots are $\frac{1}{1-\alpha}$ and $\frac{\alpha-1}{\alpha}$.

28. If α, β, γ be the roots of the equation $x^3 - 3px^2 + 3(p-1)x + 1 = 0$, find the equation whose roots are $1 - \alpha, 1 - \beta, 1 - \gamma$.

Deduce that α, β, γ are all real, if p is real.

[**Hint.** The equation whose roots are $1 - \alpha, 1 - \beta, 1 - \gamma$ has also roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$. If possible, let α, β be complex. Then $\frac{1}{\gamma} = 1 - \gamma$ and this shows that γ is also complex, a contradiction.]

5.7. Reciprocal equations.

A polynomial equation is said to be a *reciprocal equation* if the reciprocal of each of its roots is also a root of it.

Therefore a necessary condition for $f(x) = 0$ to be a reciprocal equation is that 0 is not a root of it, i.e., $f(0) \neq 0$.

Let $f(x) = 0$ be a reciprocal equation of degree n having roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let $\phi(x) = 0$ be the equation whose roots are $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$. Then the equations $f(x) = 0$ and $\phi(x) = 0$ are identical.

Let $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ be a reciprocal equation. Then it is identical with the equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$.

Therefore $(a_0, a_1, \ldots, a_n) = k(a_n, a_{n-1}, \ldots, a_0)$ for some $k \neq 0$.

That is, $a_0 = ka_n, a_1 = ka_{n-1}, a_2 = ka_{n-2}, \ldots, a_n = ka_0$.

$a_0 = ka_n$ and $a_n = ka_0$ give $k = \pm 1$.

Two cases arise.

Case I. $k = 1$.

In this case $a_0 = a_n, a_1 = a_{n-1}, \ldots, a_n = a_0$.

The coefficients of equidistant terms from the beginning and the end are equal in magnitude and have the same sign. The equation is said to be a reciprocal equation of the *first type* (or *first class*).

Case II. $k - 1$.

In this case $a_0 = -a_n, a_1 = -a_{n-1}, \ldots, a_n = -a_0$.

The coefficients of equidistant terms from the beginning and the end are equal in magnitude but of opposite signs. The equation is said to be a reciprocal equation of the *second type* (or *second class*).

Examples.

1. The equation $x^2 + 1 = 0$ is a reciprocal equation of degree 2 and of the first type.

2. The equation $3x^3 - 13x^2 + 13x - 3 = 0$ is a reciprocal equation of degree 4 and of the second type.

3. The equation $2x^4 - 5x^3 + 5x - 2 = 0$ is a reciprocal equation of degree 4 and of the second type.

4. The equation $x^4 - x^3 - x + 1 = 0$ is a reciprocal equation of degree 4 and of the first type.

5. The equation $x^4 - x^3 + x^2 + x - 1 = 0$ is not a reciprocal equation.

Theorem 5.7.1. If $f(x) = 0$ be a reciprocal equation of degree n and of the first type then $f(x) = x^n f(\frac{1}{x})$.

Conversely, if $f(x)$ be a polynomial of degree n and $f(x) = x^n f(\frac{1}{x})$ then $f(x) = 0$ is a reciprocal equation of the first type.

Proof. Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \quad a_n \neq 0.$

Since $f(x) = 0$ is a reciprocal equation of the first type,

$a_0 = a_n, a_1 = a_{n-1}, \ldots, a_n = a_0.$

Therefore $f(x) = a_0x^n + a_{n-1}x^{n-1} + \cdots + a_0$

$= x^n(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}) = x^n f(\frac{1}{x}).$

Converse part. Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n.$

Then $x^n f(\frac{1}{x}) = x^n(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + a_n)$

$= a_0 + a_1x + \cdots + a_nx^n.$

Since $f(x) = x^n f(\frac{1}{x}), a_0 = a_n, a_1 = a_{n-1}, \ldots, a_n = a_0.$

This shows that $f(x) = 0$ is a reciprocal equation of the first type.

This completes the proof.

Theorem 5.7.2. If $f(x) = 0$ be a reciprocal equation of degree n and of the second type then $f(x) = -x^n f(\frac{1}{x})$.

Conversely, if $f(x)$ be a polynomial of degree n and $f(x) = -x^n f(\frac{1}{x})$ then $f(x) = 0$ is a reciprocal equation of the second type.

Proof. Left to the reader.

Theorem 5.7.3. The solution of any reciprocal equation depends on that of a reciprocal equation of the first type and of even degree.

Proof. We are to consider three cases.

Case I. Let $f(x) = 0$ be a reciprocal equation of odd degree, say $2m+1$, and of the first type.

Let $f(x) = a_0x^{2m+1} + a_1x^{2m} + \cdots + a_{2m+1}.$

Since the equation is of the first type, $a_{2m+1} = a_0, a_{2m} = a_1, \ldots, a_{m+1} = a_m$

Therefore $f(x) = a_0(x^{2m+1}+1)+a_1x(x^{2m-1}+1)+\cdots+a_mx^m(x+1).$

$f(x)$ is divisible by $x + 1$. If $\phi(x)$ be the quotient, the solution of $f(x) = 0$ depends on the solution of $\phi(x) = 0$.

$$\phi(x) = \frac{f(x)}{x+1} = \frac{x^{2m+1}f(\frac{1}{x})}{x+1}.$$

Therefore $\phi(\frac{1}{x}) = \frac{f(x)}{x^{2m+1}(\frac{1}{x}+1)} = \frac{f(x)}{x^{2m}(x+1)} = \frac{1}{x^{2m}}\ \phi(x).$

This shows that $\phi(x) = 0$ is a reciprocal equation of degree $2m$ and of the first type.

Case II. Let $f(x) = 0$ be a reciprocal equation of odd degree, say $2m+1$, and of the second type.

Let $f(x) = a_0x^{2m+1} + a_1x^{2m} + \cdots + a_{2m+1}.$

Since the equation is of the second type

$a_{2m+1} = -a_0, a_{2m} = -a_1, \ldots, a_{m+1} = -a_m.$

Therefore $f(x) = a_0(x^{2m+1}-1)+a_1x(x^{2m-1}-1)+\cdots+a_mx^m(x-1).$

$f(x)$ is divisible by $x - 1$. If $\phi(x)$ be the quotient, the solution of $f(x) = 0$ depends on the solution of $\phi(x) = 0$.

$$\phi(x) = \frac{f(x)}{x-1} = \frac{-x^{2m+1}f(\frac{1}{x})}{x-1}.$$

Therefore $\phi(\frac{1}{x}) = \frac{f(x)}{x^{2m+1}(1-\frac{1}{x})} = \frac{f(x)}{x^{2m}(x-1)} = \frac{1}{x^{2m}}\ \phi(x).$

This shows that $\phi(x) = 0$ is a reciprocal equation of degree $2m$ and of the first type.

Case III. Let $f(x) = 0$ be a reciprocal equation of even degree, say $2m$, and of the second type.

Let $f(x) = a_0x^{2m} + a_1x^{2m-1} + \cdots + a_{2m}.$

Since the equation is of the second type, $a_{2m} = -a_0, a_{2m-1} = -a_1, \ldots, a_{m+1} = -a_{m-1}, a_m = 0.$

Hence $f(x) = a_0(x^{2m}-1)+a_1x(x^{2m-2}-1)+\cdots+a_{m-1}x^{m-1}(x^2-1).$

$f(x)$ is divisible by $x^2 - 1$. If $\phi(x)$ be the quotient, the solution of $f(x) = 0$ depends on the solution of $\phi(x) = 0$.

$$\phi(x) = \frac{f(x)}{x^2-1} = \frac{-x^{2m}f(\frac{1}{x})}{x^2-1}.$$

Therefore $\phi(\frac{1}{x}) = \frac{f(x)}{x^{2m}(1-\frac{1}{x^2})} = \frac{f(x)}{x^{2m-2}(x^2-1)} = \frac{1}{x^{2m-2}}\phi(x).$

or, $\phi(x) = x^{2m-2}\phi(\frac{1}{x}).$

This shows that $\phi(x) = 0$ is a reciprocal equation of degree $2m-2$ and of the first type.

This completes the proof.

Definition. A reciprocal equation is said to be of the *standard form* if it is of the first type and of even degree.

Theorem 5.7.4. The solution of a reciprocal equation of the first type and of degree $2m$ depends on that of an equation of degree m.

Proof. Let the equation be $a_0x^{2m} + a_1x^{2m-1} + \cdots + a_{2m} = 0$.

Since it is of the first type, $a_{2m} = a_0, a_{2m-1} = a_1, \ldots, a_{m+1} = a_{m-1}$.

Therefore the equation reduces to

$a_0(x^{2m}+1) + a_1x(x^{2m-2}+1) + \cdots + a_mx^m = 0$.

Dividing by x^m, we have

$a_0(x^m + \frac{1}{x^m}) + a_1(x^{m-1} + \frac{1}{x^{m-1}}) + \cdots + a_m = 0 \cdots \cdots$ (i)

Let $x + \frac{1}{x} = t, x^r + \frac{1}{x^r} = u_r$. Then since

$(x + \frac{1}{x})(x^{r-1} + \frac{1}{x^{r-1}}) = (x^r + \frac{1}{x^r}) + (x^{r-2} + \frac{1}{x^{r-2}})$, we have

$$u_r = tu_{r-1} - u_{r-2}.$$

But $u_0 = 2, u_1 = t$. Taking $r = 2, 3, 4, \ldots$ we have

$u_2 = t^2 - 2, \quad u_3 = t^3 - 3t, \quad u_4 = t^4 - 4t^2 + 2, \cdots \cdots$ (ii)

Using (ii), the equation (i) can be expressed as an equation in t of degree m.

Worked Examples (continued).

5. Solve the equation $6x^4 + 35x^3 + 62x^2 + 35x + 6 = 0$.

This is a reciprocal equation of the first type. This can be written as $6(x^4+1) + 35x(x^2+1) + 62x^2 = 0$.

Dividing by x^2, we have

$6(x^2 + \frac{1}{x^2}) + 35(x + \frac{1}{x}) + 62 = 0$

or, $6(t^2 - 2) + 35t + 62 = 0$ where $t = x + \frac{1}{x}$

or, $6t^2 + 35t + 50 = 0$, giving $t = -\frac{10}{3}, -\frac{5}{2}$.

When $t = -\frac{10}{3}$, we have $3x^2 + 10x + 3 = 0$, or $x = -3, -\frac{1}{3}$.

When $t = -\frac{5}{2}$, we have $2x^2 - 5x + 2 = 0$, or $x = -2, -\frac{1}{2}$.

Hence the roots are $-3, -2, -\frac{1}{3}, -\frac{1}{2}$.

6. Reduce the reciprocal equation $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$ to a reciprocal equation of the standard form and solve it.

The equation is of even degree and of the second type. This can be written as

$3(x^6 - 1) + x(x^4 - 1) - 27x^2(x^2 - 1) = 0$

or, $(x^2 - 1)\{3(x^4 + x^2 + 1) + x(x^2 + 1) - 27x^2\} = 0$

or, $(x^2 - 1)(3x^4 + x^3 - 24x^2 + x + 3) = 0$.

Either $x^2 - 1 = 0$ or, $3x^4 + x^3 - 24x^2 + x + 3 = 0$.

$x^2 - 1 = 0$ gives $x = \pm 1$.

$3x^4 + x^3 - 24x^2 + x + 3 = 0$ is a reciprocal equation of even degree and of the first type. This is of the standard form. This can be written as $3(x^4+1) + x(x^2+1) - 24x^2 = 0$.

Dividing by x^2, $3(x^2 + \frac{1}{x^2}) + (x + \frac{1}{x}) - 24 = 0$

or, $3(t^2 - 2) + t - 24 = 0$ where $t = x + \frac{1}{x}$

or, $3t^2 + t - 30 = 0$, giving $t = -\frac{10}{3}, 3$.

When $t = -\frac{10}{3}$, we have $3x^2 + 10x + 3 = 0$, or $x = -3, -\frac{1}{3}$.

When $t = 3$, we have $x^2 - 3x + 1 = 0$, or $x = \frac{3 \pm \sqrt{5}}{2}$.

Hence the roots of the given equation are $\pm 1, -3, -\frac{1}{3}, \frac{3 \pm \sqrt{5}}{2}$.

7. Prove that the equation $(x+1)^4 = a(x^4+1)$ is a reciprocal equation if $a \neq 1$ and solve it when $a = -2$.

Let $f(x) = (x+1)^4 - a(x^4+1)$.

When $a = 1, f(0) = 0$ and therefore $f(x) = 0$ is not a reciprocal equation.

When $a \neq 1, f(x)$ is a polynomial of degree 4 and

$f(\frac{1}{x}) = (\frac{1}{x} + 1)^4 - a(\frac{1}{x^4} + 1) = \frac{(1+x)^4 - a(1+x^4)}{x^4} = \frac{f(x)}{x^4}$.

Therefore $f(x) = x^4 f(\frac{1}{x})$ and this proves that $f(x) = 0$ is a reciprocal equation of degree 4.

When $a = -2$, the equation is $(x+1)^4 + 2(x^4+1) = 0$

or, $(x^2 + 2x + 1)^2 + 2(x^4 + 1) = 0$.

Dividing by x^2, we have $(x + \frac{1}{x} + 2)^2 + 2(x^2 + \frac{1}{x^2}) = 0$

or, $(t+2)^2 + 2(t^2 - 2) = 0$ where $t = x + \frac{1}{x}$

or, $3t^2 + 4t = 0$, giving $t = 0, -\frac{4}{3}$.

When $t = 0$, we have $x^2 + 1 = 0$, or $x = \pm i$.

When $t = -\frac{4}{3}$, we have $3x^2 + 4x + 3 = 0$, or $x = \frac{-2 \pm \sqrt{5}i}{3}$.

Hence the roots are $\pm i, \frac{-2 \pm \sqrt{5}i}{3}$.

8. Find a substitution of the form $x = my + n$ which will transform the equation $x^4 - 7x^3 + 13x^2 - 12x + 6 = 0$ into a reciprocal equation. Utilise this to solve the equation.

By the substituion the equation transforms to

$(my + n)^4 - 7(my + n)^3 + 13(my + n)^2 - 12(my + n) + 6 = 0$

or, $m^4y^4 + m^3y^3(4n-7) + m^2y^2(6n^2-21n+13) + my(4n^3-21n^2+26n-12) + (n^4-7n^3+13n^2-12n+6) = 0$.

This will be a reciprocal equation if $m^4 = n^4-7n^3+13n^2-12n+6$ and $m^2(4n-7) = (4n^3-21n^2+26n-12)$.

Therefore $(4n^3-21n^2+26n-12)^2 = (4n-7)^2(n^4-7n^3+13n^2-12n+6)$

or, $16n^6 - 168n^5 + 649n^4 - 1188n^3 + 1180n^2 - 624n + 144$
$= 16n^6 - 168n^5 + 649n^4 - 1263n^3 + 1405n^2 - 924n + 294$

or, $75n^3 - 225n^2 + 300n - 150 = 0$, giving $n = 1$ and $m = \pm 1$.

Taking $m = 1, n = 1$, the equation is $y^4 - 3y^3 - 2y^2 - 3y + 1 = 0$

or, $(y^4+1) - 3y(y^2+1) - 2y^2 = 0$

or, $(y^2 + \frac{1}{y^2}) - 3(y + \frac{1}{y}) - 2 = 0$

or, $t^2 - 3t - 4 = 0$, where $t = y + \frac{1}{y}$. Therefore $t = 4, -1$.

When $t = 4$, $y = 2 \pm \sqrt{3}$. When $t = -1$, $y = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Hence the roots of the given equation are $3 \pm \sqrt{3}, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Note. Taking $n = 1$, $m = -1$, the equation transforms to another reciprocal equation $y^4 + 3y^3 - 2y^2 + 3y + 1 = 0$.

9. If α, β, γ be the roots of the equation $ax^3 + bx^2 + cx + d = 0, d \neq 0$, find the equation whose roots are $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}, \gamma + \frac{1}{\gamma}$.

Since $d \neq 0$, none of α, β, γ is zero.

$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are the roots of the equation $dx^3 + cx^2 + bx + a = 0$.

Therefore $(ax^3 + bx^2 + cx + d)(dx^3 + cx^2 + bx + a)$
$= ad(x-\alpha)(x-\frac{1}{\alpha})(x-\beta)(x-\frac{1}{\beta})(x-\gamma)(x-\frac{1}{\gamma})$

or, $ad(x^6+1) + (ac+bd)(x^5+x) + (ab+bc+cd)(x^4+x^2) + (a^2+b^2+c^2+d^2)x^3 = ad\{x^2-(\alpha+\frac{1}{\alpha})x+1\}\{x^2-(\beta+\frac{1}{\beta})x+1\}\{x^2-(\gamma+\frac{1}{\gamma})x+1\}$

or, $ad(x^3+\frac{1}{x^3}) + (ac+bd)(x^2+\frac{1}{x^2}) + (ab+bc+cd)(x+\frac{1}{x}) + (a^2+b^2+c^2+d^2) = ad\{(x+\frac{1}{x})-(\alpha+\frac{1}{\alpha})\}\{(x+\frac{1}{x})-(\beta+\frac{1}{\beta})\}\{(x+\frac{1}{x})-(\gamma+\frac{1}{\gamma})\}$.

Let $x + \frac{1}{x} = t$. Then
$ad(t^3-3t) + (ac+bd)(t^2-2) + (ab+bc+cd)t + (a^2+b^2+c^2+d^2)$
$= ad\{t-(\alpha+\frac{1}{\alpha})\}\{t-(\beta+\frac{1}{\beta})\}\{t-(\gamma+\frac{1}{\gamma})\}$.

Hence the required equation is
$adt^3 + (ac+bd)t^2 + (ab+bc+cd-3ad)t + (a-c)^2 + (b-d)^2 = 0$.

5.8. Binomial equations.

The binomial equation $x^n - 1 = 0$ has n roots $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, where $k = 0, 1, 2, \ldots, n-1$.

They are called the nth *roots of unity*. The roots can be expressed as $(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})^k$, where $k = 0, 1, 2, \ldots, n-1$
or, as $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$, where $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

If n be an odd integer, the equation has only one real root, by Descartes' rule of signs. The real root is 1. Therefore all the roots $\alpha, \alpha^2, \ldots, \alpha^{n-1}$ are imaginary.
Since $\alpha^n = 1$, we have $\alpha^{n-1} = \frac{1}{\alpha}, \alpha^{n-2} = \frac{1}{\alpha^2}, \cdots, \alpha^{(n+1)/2} = \frac{1}{\alpha^{(n-1)/2}}$.

Therefore the imaginary roots are reciprocal pairs $(\alpha, \frac{1}{\alpha})$, $(\alpha^2, \frac{1}{\alpha^2})$, $\ldots, (\alpha^{(n-1)/2}, \frac{1}{\alpha^{(n-1)/2}})$ and they are the roots of the reciprocal equation
$x^{n-1} + x^{n-2} + \cdots + x + 1 = 0$.

If n be an even integer, the equation has only one positive root and only one negative root by Descartes'rule of signs.
They are 1 and -1. All other roots are imaginary and they are $\alpha, \alpha^2, \ldots, \alpha^{n/2-1}, \alpha^{n/2+1}, \ldots, \alpha^{n-1}$.

These can be arranged in reciprocal pairs $(\alpha, \frac{1}{\alpha})$, $(\alpha^2, \frac{1}{\alpha^2})$, $\cdots$, $(\alpha^{n/2-1}, \frac{1}{\alpha^{n/2-1}})$; and they are the roots of the reciprocal equation
$x^{n-2} + x^{n-4} + \cdots + x^2 + 1 = 0$.

Let a be a non zero complex number and $a = r(\cos\theta + i\sin\theta)$, $-\pi < \theta \leq \pi$.

The binomial equation $x^n - a = 0$ has n roots $\sqrt[n]{r}(\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n})$, where $k = 0, 1, 2, \ldots, n-1$.

The roots can be expressed as $\sqrt[n]{r}(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n})(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$, where $k = 0, 1, 2, \ldots, n-1$
or, as $\alpha_0.\alpha^k$, where $\alpha_0 = \sqrt[n]{r}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$ and $\alpha = (\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})$.

α_0 is called the *principal* nth root of a.
Therefore all the n roots of the equation are obtained by multiplying the principal nth root of a with all the nth roots of unity.

Properties.

I. If α be any root of $x^n - 1 = 0$, then α^m is also a root, where m is an integer.

Proof. Since α is a root, $\alpha^n = 1$. Therefore $(\alpha^m)^n = \alpha^{mn} = (\alpha^n)^m = 1$.

This proves that α^m is a root of $x^n = 1$.

II. If m, n are integers prime to each other, the equations $x^m - 1 = 0$ and $x^n - 1 = 0$ have no common root except 1.

Proof. Since m and n are prime to each other, there exist integers a and b such that $am + bn = 1$.

Let α be a common root of $x^m - 1 = 0$ and $x^n - 1 = 0$.

Then $\alpha^m = 1$ and $\alpha^n = 1$.

Therefore $\alpha = \alpha^{am+bn} = (\alpha^m)^a.(\alpha^n)^b = 1$.

This proves that 1 is the only common root.

III. If d be the g.c.d. of m and n, the common roots of $x^m - 1 = 0$ and $x^n - 1 = 0$ are roots of $x^d - 1 = 0$ and conversely, all the roots of $x^d - 1 = 0$ are common roots of $x^m - 1 = 0$ and $x^n - 1 = 0$.

Proof. Since d is the g.c.d. of m and n there exist integers m', n' prime to each other such that $m = dm', n = dn'$.

Since m' and n' are prime to each other, there exist integers a and b such that $am' + bn' = 1$.

Let α be a common root of $x^m - 1 = 0$ and $x^n - 1 = 0$.

Then $\alpha^m = 1$ and $\alpha^n = 1$. Therefore $\alpha^d = \alpha^{d(am'+bn')} = (\alpha^{dbn'})^a.(\alpha^{dn'})^b = 1$.

This proves that α is a root of $x^d - 1 = 0$.

Conversely, let β be a root of $x^d - 1 = 0$. Then $\beta^d = 1$.

Therefore $\beta^{dm'} = 1$ and $\beta^{dn'} = 1$.

That is, $\beta^m = 1$ and $\beta^n = 1$.

Hence β is a common root of $x^m - 1 = 0$ and $x^n - 1 = 0$. □

IV. If n be a prime and α is an imaginary root of $x^n - 1 = 0$ then $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ is a complete list of the roots of the equation $x^n - 1 = 0$.

Proof. First we prove that no two of $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are equal.

If possible, let $\alpha^p = \alpha^q$ where $0 \le q < p \le n - 1$. Then $\alpha^{p-q} = 1$.

This shows that the equations $x^n - 1 = 0$ and $x^{p-q} - 1 = 0$ have a common imaginary root α. But since n is a prime and $p - q < n, n$ is prime to $p - q$ and therefore the equations $x^n - 1 = 0$ and $x^{p-q} - 1 = 0$ cannot have a common root other than 1.

Thus we arrive at a contradiction and therefore no two of $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are equal.

Again each of $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ is a root of $x^n - 1 = 0$. Since these roots are n in number, $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ is a complete list of the roots of $x^n - 1 = 0$.

Note. If n be a composite number and α is an imaginary root of the equation $x^n - 1 = 0$, then $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ may not give a complete list of the roots of $x^n - 1 = 0$.

For example, $\cos \frac{4\pi}{8} + i \sin \frac{4\pi}{8} = i$ is an imaginary root of $x^8 - 1 = 0$ but $1, i, i^2, \ldots, i^7$ do not give a complete list of roots of $x^8 - 1 = 0$.

On the other hand, $\beta = \cos \frac{6\pi}{8} + i \sin \frac{6\pi}{8}$ is an imaginary root of $x^8 - 1 = 0$ and $1, \beta, \beta^2, \ldots, \beta^7$ give a complete list of roots of $x^8 - 1 = 0$.

Therefore if n be a composite number, the theorem is true for some imaginary roots of $x^n - 1 = 0$ but not for all imaginary roots. Such an imaginary root for which the theorem is true, whether n is a prime or a composite number, is called a *special root* of $x^n - 1 = 0$.

Worked Examples.

1. If α be an imaginary root of $x^n - 1 = 0$ where n is a prime number, prove that $(1 - \alpha)(1 - \alpha^2) \ldots (1 - \alpha^{n-1}) = n$.

Since n is a prime, all the roots of $x^n - 1 = 0$ are given by $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$.

Therefore $x^n - 1 = (x - 1)(x - \alpha)(x - \alpha^2) \ldots (x - \alpha^{n-1})$

or, $x^{n-1} + x^{n-2} + \cdots + x + 1 = (x - \alpha)(x - \alpha^2) \ldots (x - \alpha^{n-1})$.

When $x = 1$, we have $n = (1 - \alpha)(1 - \alpha^2) \ldots (1 - \alpha^{n-1})$.

2. If α be an imaginary root of the equation $x^7 - 1 = 0$, find the equation whose roots are $\alpha + \alpha^6, \alpha^2 + \alpha^5, \alpha^3 + \alpha^4$.

Since 7 is a prime and α is an imaginary root of $x^7 - 1 = 0$, all the roots of $x^7 - 1 = 0$ are given by $1, \alpha, \alpha^2, \ldots, \alpha^6$.

Therefore $x^7 - 1 = (x - 1)(x - \alpha)(x - \alpha^2) \ldots (x - \alpha^6)$

or, $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

$= (x - \alpha)(x - \alpha^6)(x - \alpha^2)(x - \alpha^5)(x - \alpha^3)(x - \alpha^4)$

$= \{x^2 - (\alpha + \alpha^6)x + 1\}\{x^2 - (\alpha^2 + \alpha^5)x + 1\}\{x^2 - (\alpha^3 + \alpha^4)x + 1\}$.

Dividing by x^3, we have

$(x^3 + \frac{1}{x^3}) + (x^2 + \frac{1}{x^2}) + (x + \frac{1}{x}) + 1$

$= \{x - (\alpha + \alpha^6) + \frac{1}{x})\}\{x - (\alpha^2 + \alpha^5) + \frac{1}{x})\}\{x - (\alpha^3 + \alpha^4) + \frac{1}{x})\}$.

Let $x + \frac{1}{x} = t$. Then

$t^3 + t^2 - 2t - 1 = \{t - (\alpha + \alpha^6)\}\{t - (\alpha^2 + \alpha^5)\}\{t - (\alpha^3 + \alpha^4)\}$.

Hence the equation whose roots are $\alpha + \alpha^6, \alpha^2 + \alpha^5, \alpha^3 + \alpha^4$ is $x^3 + x^2 - 2x - 1 = 0$.

3. If α be an imaginary root of the equation $x^5 - 1 = 0$, find the equation whose roots are $\alpha + 2\alpha^4, \alpha^2 + 2\alpha^3, \alpha^3 + 2\alpha^2, \alpha^4 + 2\alpha$.

Since 5 is a prime and α is an imaginary root of $x^5 - 1 = 0$, all the roots of $x^5 - 1 = 0$ are given by $1, \alpha, \alpha^2, \ldots, \alpha^4$.

Therefore $\alpha, \alpha^2, \alpha^3, \alpha^4$ are the roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$ and $2\alpha, 2\alpha^2, 2\alpha^3, 2\alpha^4$ are the roots of the equation $x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$.

Therefore $(x^4 + x^3 + x^2 + x + 1)(x^4 + 2x^3 + 4x^2 + 8x + 16)$
$= (x - \alpha)(x - 2\alpha^4)(x - \alpha^2)(x - 2\alpha^3)(x - \alpha^3)(x - 2\alpha^2)(x - \alpha^4)(x - 2\alpha)$

or, $x^8 + 3x^7 + 7x^6 + 15x^5 + 31x^4 + 30x^3 + 28x^2 + 24x + 16$
$= \{x^2 - (\alpha + 2\alpha^4)x + 2\}\{x^2 - (\alpha^2 + 2\alpha^3)x + 2\}\{x^2 - (\alpha^3 + 2\alpha^2)x + 2\}\{x^2 - (\alpha^4 + 2\alpha)x + 2\}$

or, $(x^4 + \frac{16}{x^4}) + 3(x^3 + \frac{8}{x^3}) + 7(x^2 + \frac{4}{x^2}) + 15(x + \frac{2}{x}) + 31$
$= \{(x + \frac{2}{x} - (\alpha + 2\alpha^4)\} \cdots \{(x + \frac{2}{x} - (\alpha^4 + 2\alpha)\}$.

Let $x + \frac{2}{x} = t$. Then $x^2 + \frac{4}{x^2} = t^2 - 4, x^3 + \frac{8}{x^3} = t^3 - 6t, x^4 + \frac{16}{x^4} = t^4 - 8t^2 + 8$.

Therefore $t^4 + 3t^3 - t^2 - 3t + 11 = \{t - (\alpha + 2\alpha^4)\} \cdots \{t - (\alpha^4 + 2\alpha)\}$.

Hence the required equation is $x^4 + 3x^3 - x^2 - 3x + 11 = 0$.

4. Solve the equation $x^5 - 1 = 0$ and deduce the values of $\cos\frac{\pi}{5}$ and $\cos\frac{2\pi}{5}$.

The roots of $x^5 - 1 = 0$ are $\cos\frac{2k\pi}{5} + i \sin\frac{2k\pi}{5}, k = 0, 1, 2, 3, 4$.

i.e., $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ where $\alpha = \cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}$.

As $\alpha^5 = 1$, the roots are $1, \alpha, \alpha^2, \frac{1}{\alpha^2}, \frac{1}{\alpha}$.

Therefore $x^4 + x^3 + x^2 + x + 1 = (x - \alpha)(x - \frac{1}{\alpha})(x - \alpha^2)(x - \frac{1}{\alpha^2})$
$= \{x^2 - (\alpha + \frac{1}{\alpha})x + 1\}\{x^2 - (\alpha^2 + \frac{1}{\alpha^2})x + 1\}$.

Dividing by x^2, we have
$(x^2 + \frac{1}{x^2}) + (x + \frac{1}{x}) + 1 = \{(x + \frac{1}{x}) - (\alpha + \frac{1}{\alpha})\}\{(x + \frac{1}{x}) - (\alpha^2 + \frac{1}{\alpha^2})\}$

Let $x + \frac{1}{x} = t$. Then
$t^2 + t - 1 = (t - 2\cos\frac{2\pi}{5})(t - 2\cos\frac{4\pi}{5})$.

This shows that $2\cos\frac{2\pi}{5}, 2\cos\frac{4\pi}{5}$ are the roots of the equation $t^2 + t - 1 = 0$. The roots of the equation are $\frac{\sqrt{5}-1}{4}, \frac{-\sqrt{5}-1}{4}$.

Therefore $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}, \cos\frac{4\pi}{5} = -\frac{\sqrt{5}+1}{4}$, since $\cos\frac{2\pi}{5} > 0, \cos\frac{4\pi}{5} < 0$. Hence $\cos\frac{\pi}{5} = -\cos\frac{4\pi}{5} = \frac{\sqrt{5}+1}{4}$ and $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$.

5.9. Special roots of the equation $x^n - 1 = 0$.

A root of the equation $x^n - 1 = 0$ which is not a root of the equation $x^m - 1 = 0$, where m is any integer less than n, is said to be a *special root* of the equation $x^n - 1 = 0$.

Theorem 5.9.1. The special roots of the equation $x^n - 1 = 0$ are $\cos\frac{2r\pi}{n} + i \sin\frac{2r\pi}{n}$, where r is a positive integer less n and prime to n.

Proof. The roots of $x^n - 1 = 0$ are $1, \cos\frac{2r\pi}{n} + i \sin\frac{2r\pi}{n}$, where $0 < r < n$.

1 cannot be a special root. The special root, if there be any, must be one or more of $\cos\frac{2r\pi}{n} + i \sin\frac{2r\pi}{n}$, where $0 < r < n$.

Let us consider the following cases.

Case **I.** r is not prime to n. Let $\gcd(r, n) = d$. Then $d > 1$ and there exist integers r', n' such that $r = dr', n = dn'$. Clearly, $n' < n$.

$$\left(\cos\tfrac{2r\pi}{n} + i \sin\tfrac{2r\pi}{n}\right)^{n'} = \cos\tfrac{2rn'\pi}{n} + i \sin\tfrac{2rn'\pi}{n}$$
$$= \cos\tfrac{2dr'n'\pi}{n} + i \sin\tfrac{2dr'n'\pi}{n}$$
$$= \cos 2r'\pi + i \sin 2r'\pi = 1.$$

This shows that $\cos\frac{2r\pi}{n} + i \sin\frac{2r\pi}{n}$ is a root of the equation $x^{n'} - 1 = 0$ where $n' < n$ and hence $\cos\frac{2r\pi}{n} + i \sin\frac{2r\pi}{n}$ cannot be a special root of the equation $x^n - 1 = 0$.

Case **II.** r is prime to n. We prove that $\cos\frac{2r\pi}{n} + i \sin\frac{2r\pi}{n}$ cannot be a root of the equation $x^m - 1 = 0$, where m is any positive integer less than n.

If possible, let $(\cos\frac{2r\pi}{n} + i \sin\frac{2r\pi}{n})^m = 1$ where $0 < m < n$.

Then $\cos\frac{2rm\pi}{n} + i \sin\frac{2mr\pi}{n} = 1$.

This requires rm must be divisible by n. But since r is prime to n, m must be divisible by n which cannot happen since $m < n$.

This proves that $\cos\frac{2r\pi}{n} + i \sin\frac{2r\pi}{n}$ is a special root of $x^n - 1 = 0$.

Note 1. If n be a prime number, every root of the equation $x^n - 1 = 0$ except 1 is a special root.

Note 2. A non-special root of the equation $x^n - 1 = 0$ is a root of the equation $x^m - 1 = 0$ for some $m < n$. Since the common roots of the equations $x^n - 1 = 0$ and $x^m - 1 = 0$ are the roots of $x^d - 1 = 0$, where $d = gcd(m, n)$, (i.e., d is a divisor of n), it follows that each non-special root of the equation $x^n - 1 = 0$ is a root of the equation $x^d - 1 = 0$ for some divisor $d(< n)$ of n.

For example, the non-special roots of the equation $x^9 - 1 = 0$ are given by the roots of the equations $x - 1 = 0$ and $x^3 - 1 = 0$, since 1 and 3 are the only divisors of 9 and less than 9.

Note 3. The number of special roots of the equation $x^n - 1 = 0$ is the number of positive integers less than n and prime to n. Thus the number of the special roots of the equation $x^n - 1 = 0$ is $\phi(n)$, where ϕ is the Euler's phi function.

Theorem 5.9.2. If α be a special root of the equation $x^n - 1 = 0$, then $\frac{1}{\alpha}$ is also a special root of it.

Proof. The special roots of the equation $x^n - 1 = 0$ are $\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$, where r is less than n and prime to n.

If r is less than n and prime to n, then $n - r$ is also less than n and prime to n. Therefore if $\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$ be a special root, then

$\cos\frac{2(n-r)\pi}{n} + i\sin\frac{2(n-r)\pi}{n} = \cos\frac{2r\pi}{n} - i\sin\frac{2r\pi}{n} = \frac{1}{\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}}$ is also a special root.

This shows that if α be a special root of the equation $x^n - 1 = 0$, then $\frac{1}{\alpha}$ is also a special root of it.

Note. The special root of the equation $x^n - 1 = 0$ are the roots of a reciprocal equation of degree $\phi(n)$.

Theorem 5.9.3. If α be a special root of the equation $x^n - 1 = 0$, then $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ is a complete list of distinct roots of the equation $x^n - 1 = 0$.

Proof. First we prove that no two of $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are equal.

If possible, let $\alpha^p = \alpha^q$ where $0 \le p \le n-1, 0 \le q \le n-1$ and $p > q$. Then $\alpha^{p-q} = 1$.

This shows that α is a root of the equation $x^{p-q} - 1 = 0$ whose degree is less than n and consequently, α is not a special root of the equation $x^n - 1 = 0$, a contradiction. Therefore no two of $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are equal.

Again each of $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ is a root of $x^n - 1 = 0$. Since these roots are n in number, $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ is a complete list of the roots of the equation $x^n - 1 = 0$.

Note. If n be a prime and α be a special root of the equation $x^n - 1 = 0$, then $\alpha, \alpha^2, \ldots, \alpha^{n-1}$ is a complete list of special roots of the equation $x^n - 1 = 0$.

Theorem 5.9.4. If α be a special root of the equation $x^n - 1 = 0$, the complete set of special roots is given by $\{\alpha^a, \alpha^b, \ldots\}$ where $a, b, \ldots$ are integers less than n and prime to n.

Proof. Each of $\alpha^a, \alpha^b, \ldots$ is a special root by the previous theorem. No two of them are equal, because $\alpha^a = \alpha^b \Rightarrow \alpha^{a-b} = 1$ and since $a - b < n$, this implies that α is not a special root.

We now prove that each special root of $x^n - 1 = 0$ is included in the set $\{\alpha^a, \alpha^b, \ldots\}$.

Let β be any special root. Then β being a root of $x^n - 1 = 0$ is one of $\alpha, \alpha^2, \ldots, \alpha^{n-1}$.

If possible, let $\beta = \alpha^s$ where s is less than n and not prime to n. Let d be the g.c.d. of s and n. Then there exist integers s', n' such that $s = ds', n = dn'$ and $n' < n$.

Now $\beta^{n'} = \alpha^{sn'} = \alpha^{ds'n} = (\alpha^n)^{s'} = 1$, a contradiction, since β is a special root. Therefore β is one of $\alpha^a, \alpha^b, \ldots$ and the proof is complete.

Theorem 5.9.5. If $n = pqr$ where p, q, r are distinct primes or powers of distinct primes, the roots of $x^n - 1 = 0$ are n terms of the product $(1+\alpha+\alpha^2+\cdots+\alpha^{p-1})(1+\beta+\beta^2+\cdots+\beta^{q-1})(1+\gamma+\gamma^2+\cdots+\gamma^{r-1})$, where α is a special root of $x^p - 1 = 0$, β is a special root of $x^q - 1 = 0$, γ is a special root of $x^r - 1 = 0$.

Proof. Since α is a special root of the equation $x^p - 1 = 0$, $1, \alpha, \alpha^2, \ldots, \alpha^{p-1}$ is a complete list of distinct roots of the equation $x^p - 1 = 0$. Similarly, $1, \beta, \beta^2, \ldots, \beta^{q-1}$ is a complete list of the roots of the equation $x^q - 1 = 0$; $1, \gamma, \gamma^2, \ldots, \gamma^{r-1}$ is a complete list of the roots of the equation $x^r - 1 = 0$.

Any term of the product is of the form $\alpha^a\beta^b\gamma^c$ where $0 \le a \le p-1, 0 \le b \le q-1, 0 \le c \le r-1$.

$\alpha^a\beta^b\gamma^c$ is a root of $x^n - 1 = 0$ because $(\alpha^a\beta^b\gamma^c)^n = \alpha^{an}\beta^{bn}\gamma^{cn} = (\alpha^p)^{aqr}(\beta^q)^{brp}(\gamma^r)^{cpq} = 1$.

The n terms of the product will give a complete list of roots of $x^n - 1 = 0$ if we can prove that no two terms of the product are equal.

If possible, let $\alpha^a\beta^b\gamma^c = \alpha^{a'}\beta^{b'}\gamma^{c'}$, where

$$0 \le a \le p-1, 0 \le b \le q-1, 0 \le c \le r-1$$

$$0 \le a' \le p-1, 0 \le b' \le q-1, 0 \le c' \le r-1.$$

Then $\alpha^{a-a'} = \beta^{b'-b}\gamma^{c'-c}$.

But $\alpha^{a-a'}$ is a root of $x^p - 1$ and $\beta^{b'-b}\gamma^{c'-c}$ is a root of $x^{qr} - 1 = 0$ and since p and qr are prime to each other, the equations $x^p - 1 = 0$ and

$x^{qr} - 1 = 0$ have no common root except 1.

Since α is a special root, $\alpha \neq 1$ and also $\alpha^{a-a'} \neq 1$, unless $a = a'$ or $a - a'$ is an integral multiple of p.

Therefore $\alpha^a\beta^b\gamma^c = \alpha^{a'}\beta^{b'}\gamma^{c'}$ can occur only when $a = a', b = b'$ and $c = c'$.

This proves that no two terms of the product are equal and therefore n terms of the product give a complete ltst of distinct roots of the equation $x^n - 1 = 0$.

Note. The theorem can be generalised to the case when n is the product of a finite number of distinct primes or powers of a finite number of distinct primes.

Examples. (i) Let $n = 6 = 2.3$. -1 is a special root of $x^2 - 1 = 0$, ω is a special root of $x^3 - 1 = 0$.

Then the 6 terms of the product $[1 + (-1)][1 + \omega + \omega^2]$ give all the roots of the equation $x^6 - 1 = 0$. The roots are $1, -1, \omega, -\omega, \omega^2, -\omega^2$.

(ii) Let $n = 12 = 2^2.3$. i is a special root of $x^4 - 1 = 0$, ω is a special root of $x^3 - 1 = 0$.

Then the 12 terms of the product $[1 + i + i^2 + i^3][1 + \omega + \omega^2]$ give all the roots of the equation $x^{12} - 1 = 0$. The roots are $1, \omega, \omega^2, i, i\omega, i\omega^2, -1, -\omega, -\omega^2, -i, -i\omega, -i\omega^2$.

Theorem 5.9.6. Let $n = pq$, where p and q are prime to each other. If α be a special root of the equation $x^p - 1 = 0$ and β be a special root of the equation $x^q - 1 = 0$, then $\alpha\beta$ is a special root of the equation $x^n - 1 = 0$.

Proof. $(\alpha\beta)^n = (\alpha\beta)^{pq} = 1$ and therefore $\alpha\beta$ is a root of the equation $x^n - 1 = 0$.

Let $(\alpha\beta)^k = 1$. Then $\alpha^k = \beta^{-k}$. Therefore $\alpha^{qk} = \beta^{-qk}$.

β being a root of $x^q - 1 = 0, \beta^q = 1$ and therefore $\alpha^{qk} = 1$.

Since α is a special root of $x^p - 1 = 0$ and $\alpha^{qk} = 1$, it follows that $qk > p$.

Let $qk = sp + r$ where s and r are integers and $0 \leq r < p$.
$1 = \alpha^{qk} \Rightarrow 1 = \alpha^{sp+r} \Rightarrow 1 = \alpha^r$.
Therefore $r = 0$ and hence qk is divisible by p.

Since p and q are prime to each other, qk is divisible by p implies k is divisible by p.

By similar arguments, k is divisible by q.

Since p and q are prime to each other, k is divisible by n.
Therefore $(\alpha\beta)^k = 1 \Rightarrow k$ is divisible by n. This proves that $\alpha\beta$ is a special root of the equation $x^n - 1 = 0$.

Note 1. The theorem can be generalised.

Let $n = p_1p_2...p_r$, where $p_1, p_2, ..., p_r$ are pairwise relatively primes or powers of pairwise relatively primes. If α_1 be a special root of $x^{p_1} - 1 = 0$, α_2 be a special root of $x^{p_2} - 1 = 0, ..., \alpha_r$ be a special root of $x^{p_r} - 1 = 0$ then $\alpha_1\alpha_2 \ldots \alpha_r$ is a special root of the equation $x^n - 1 = 0$.

Note 2. If p and q are relatively prime, $\phi(pq) = \phi(p).\phi(q)$. Therefore the number of special roots of $x^{pq} - 1 = 0$ are obtained by multiplying each of the $\phi(p)$ special root of $x^p - 1 = 0$ by each of the $\phi(q)$ special root of $x^q - 1 = 0$.

In particular, let $n = pq$, where p, q are distinct primes.

If α be a special root of $x^p - 1 = 0$ and β be a special root of $x^q - 1 = 0$, then the special roots of $x^n - 1 = 0$ are $(p-1)(q-1)$ terms in the product $(\alpha + \alpha^2 + \cdots + \alpha^{p-1})(\beta + \beta^2 + \cdots + \beta^{q-1})$, since each term in the product is a special root of $x^{pq} - 1 = 0$ and there are $\phi(pq)$ terms in the product.

Examples.

1. $6 = 2.3$. 2 and 3 are relatively prime. The special roots of $x^3 - 1 = 0$ are ω, ω^2; and the only special root of $x^2 - 1 = 0$ is -1.

Hence the special roots of $x^6 - 1 = 0$ are $-\omega, -\omega^2$, i.e., $\frac{1 \pm \sqrt{3}i}{2}$.

Note. The special roots of $x^3 - 1 = 0$ are all non-real roots of $x^3 - 1 = 0$. Hence the special roots of $x^6 - 1 = 0$ are negative of all the non-real roots of the equation $x^3 - 1 = 0$. But the equation whose roots are negative of the roots of the equation $x^3 - 1 = 0$ is $x^3 + 1 = 0$.
Therefore the special roots of $x^6 - 1 = 0$ are all the non-real roots of the equation $x^3 + 1 = 0$.

The number of special roots of $x^n - 1 = 0$.

1. If n be a prime, the number of integers less than n and prime to n is $n - 1$. Therefore the number of special roots of $x^n - 1 = 0$ is $n - 1$.

2. If $n = p^\alpha$, where p is a prime and α is a positive integer > 1, the integers $\leq p^\alpha$ and not prime to p^α are $p, 2p, 3p, \ldots, p^{\alpha-1}p$. Therefore the number of integers less than p^α and prime to p^α is $p^\alpha - p^{\alpha-1} = p^\alpha(1 - \frac{1}{p}) = n(1 - \frac{1}{p})$.

So the number of special roots of $x^n - 1 = 0$ is $n(1 - \frac{1}{p})$.

3. If $n = p^\alpha q^\beta$, where p, q are distinct primes.

Let $\phi(m)$ denote the number of integers less than m and prime to m. Then $\phi(mn) = \phi(m)\phi(n)$, where m, n are prime to each other.

Since p, q are distinct primes, p^α, q^β are prime to each other. Therefore $\phi(p^\alpha q^\beta) = \phi(p^\alpha)\phi(q^\beta)$.

$\phi(n) = \phi(p^\alpha)\phi(q^\beta) = p^\alpha(1 - \frac{1}{p})q^\beta(1 - \frac{1}{q}) = n(1 - \frac{1}{p})(1 - \frac{1}{q})$.

Therefore the number of special roots is $n(1 - \frac{1}{p})(1 - \frac{1}{q})$.

4. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $p_1, p_2, \dots, p_k$ are distinct primes, the number of special roots of $x^n - 1 = 0$ is $n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_k})$.

For example, the number of special roots of the equation $x^{40} - 1 = 0$ is $\phi(40)$ and $\phi(40) = \phi(2^3.5) = 40(1 - \frac{1}{2})(1 - \frac{1}{5}) = 16$.

Worked Examples.

1. Find the special roots of the equation $x^{12} - 1 = 0$.

The special roots of $x^{12} - 1 = 0$ are $\cos \frac{2r\pi}{12} + i \sin \frac{2r\pi}{12}$, where r is less than 12 and prime to 12.

The integers less than 12 and prime to 12 are $1, 5, 7, 11$.

Thus the special roots are $\cos \frac{2r\pi}{12} \pm i \sin \frac{2r\pi}{12}$, where $r = 1, 5$.

i.e., $\cos \frac{\pi}{6} \pm i \sin \frac{\pi}{6}, \cos \frac{5\pi}{6} \pm i \sin \frac{5\pi}{6}$, i.e., $\frac{\sqrt{3} \pm i}{2}, \frac{-\sqrt{3} \pm i}{2}$.

Alternative method.

Since $1, 2, 3, 4$ and 6 are the only divisors of 12 and less than 12, the non-special roots of the equation $x^{12} - 1 = 0$ are the roots of $x - 1 = 0, x^2 - 1 = 0, x^3 - 1 = 0, x^4 - 1 = 0$ and $x^6 - 1 = 0$.

The l.c.m. of $x - 1, x^2 - 1, x^3 - 1, x^6 - 1$ and $x^4 - 1$ is $(x^6 - 1).(x^2 + 1)$.

$\frac{x^{12} - 1}{(x^6 - 1)(x^2 + 1)} = x^4 - x^2 + 1$.

The special roots of $x^{12} - 1 = 0$ are the roots of $x^4 - x^2 + 1 = 0$.

Dividing by x^2, we have $(x^2 + \frac{1}{x^2}) - 1 = 0$.

Let $t = x + \frac{1}{x}$. Then $t^2 - 3 = 0$. This gives $t = \pm\sqrt{3}$.

When $t = \sqrt{3}, x = \frac{\sqrt{3} \pm i}{2}$. When $t = -\sqrt{3}, x = \frac{-\sqrt{3} \pm i}{2}$.

Hence the special roots are $\frac{\sqrt{3} \pm i}{2}, \frac{-\sqrt{3} \pm i}{2}$.

2. Show that the special roots of the equation $x^9 - 1 = 0$ are the roots of the equation $x^6 + x^3 + 1 = 0$.

Since 1 and 3 are the only divisors of 9 and less than 9, the non-special roots of the equation $x^9 - 1 = 0$ are given by the roots of $x - 1 = 0$ and $x^3 - 1 = 0$.

The l.c.m. of $x - 1$ and $x^3 - 1$ is $x^3 - 1$. $\frac{x^9 - 1}{x^3 - 1} = x^6 + x^3 + 1$.

The special roots are the roots of the equation $x^6 + x^3 + 1 = 0$.

3. Find the special roots of the equation $x^{24} - 1 = 0$. Deduce the values of $\cos \frac{\pi}{12}$ and $\cos \frac{5\pi}{12}$.

The special roots of $x^{24} - 1 = 0$ are $\cos \frac{2r\pi}{24} + i \sin \frac{2r\pi}{24}$ where r is less than 24 and prime to 24, i.e., when $r = 1, 5, 7, 11, 13, 17, 19, 23$.

Let $\alpha = \cos \frac{\pi}{12} + i \sin \frac{\pi}{12}$. Then the special roots are
$\alpha, \alpha^5, \alpha^7, \alpha^{11}, \frac{1}{\alpha^{11}}, \frac{1}{\alpha^7}, \frac{1}{\alpha^5}, \frac{1}{\alpha}$.

The divisors of 24 (< 24) are 1,2,3,4,6,8 and 12. The non-special roots of $x^{24} - 1 = 0$ are the roots of $x^{12} - 1 = 0$ and $x^8 - 1 = 0$, since the divisors $1, 2, 3, 4, 6, 12$ of 24 are also divisors of 12.

The l.c.m. of $x^{12} - 1$ and $x^8 - 1$ is $(x^{12} - 1)(x^4 + 1)$.

$\frac{x^{24} - 1}{(x^{12} - 1)(x^4 + 1)} = x^8 - x^4 + 1$.

The special roots are the roots of the equation $x^8 - x^4 + 1 = 0$.

Therefore $x^8 - x^4 + 1 = [(x - \alpha)(x - \frac{1}{\alpha})][(x - \alpha^5)(x - \frac{1}{\alpha^5})]$
$[(x - \alpha^7)(x - \frac{1}{\alpha^7})][(x - \alpha)(x - \frac{1}{\alpha^{11}})]$

$= [x^2 - (\alpha + \frac{1}{\alpha})x + 1][x^2 - (\alpha^5 + \frac{1}{\alpha^5})x + 1][x^2 - (\alpha^7 + \frac{1}{\alpha^7})x + 1]$
$[x^2 - (\alpha^{11} + \frac{1}{\alpha^{11}})x + 1]$.

Dividing by x^4, we have

$x^4 + \frac{1}{x^4} - 1 = [(x + \frac{1}{x}) - 2\cos \frac{\pi}{12}][(x + \frac{1}{x}) - 2\cos \frac{5\pi}{12}]$
$[(x + \frac{1}{x}) - 2\cos \frac{7\pi}{12}][(x + \frac{1}{x}) - 2\cos \frac{11\pi}{12}]$.

Let $x + \frac{1}{x} = t$. Then

$t^4 - 4t^2 + 1 = (t - 2\cos \frac{\pi}{12})(t - 2\cos \frac{5\pi}{12})(t - 2\cos \frac{7\pi}{12})(t - 2\cos \frac{11\pi}{12})$
$= (t^2 - 4\cos^2 \frac{\pi}{12})(t^2 - 4\cos^2 \frac{5\pi}{12})$.

Taking $t^2 = z, z^2 - 4z + 1 = (z - 4\cos^2 \frac{\pi}{12})(z - 4\cos^2 \frac{5\pi}{12})$.

This shows that $4\cos^2 \frac{\pi}{12}, 4\cos^2 \frac{5\pi}{12}$ are the roots of $z^2 - 4z + 1 = 0$.

The roots of the equation $z^2 - 4z + 1 = 0$ are $2 \pm \sqrt{3}$.

Hence $\cos^2 \frac{\pi}{12} = \frac{2 + \sqrt{3}}{4}, \cos^2 \frac{5\pi}{12} = \frac{2 - \sqrt{3}}{4}$ since $\cos^2 \frac{\pi}{12} > \cos^2 \frac{5\pi}{12}$.

Therefore $\cos \frac{\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}, \cos \frac{5\pi}{12} = \frac{\sqrt{2 - \sqrt{3}}}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$.

Exercises 5E

1. Solve the reciprocal equations

(i) $x^4 + x^3 + 2x^2 + x + 1 = 0$,

(ii) $x^4 - 8x^3 + 17x^2 - 8x + 1 = 0$,

(iii) $x^4 - 4x^3 + 3x^2 - 4x + 1 = 0$,

(iv) $2x^5 + 5x^4 - 5x - 2 = 0$,

(v) $3x^5 + 7x^4 - 4x^3 + 4x^2 - 7x - 3 = 0$,

(vi) $2x^5 - 3x^4 - x^3 - x^2 - 3x + 2 = 0$,

(vii) $x^6 - 8x^4 + 8x^2 - 1 = 0$,

(viii) $x^6 + 8x^4 + 8x^2 + 1 = 0$,

(ix) $x^6 - x^5 + x^4 - 2x^3 + x^2 - x + 1 = 0$,

(x) $x^7 + 4x^6 + 4x^5 + x^4 - x^3 - 4x^2 - 4x - 1 = 0$.

2. Prove that each equation is reciprocal and solve it.

(i) $(x+1)^4 + x^4 + 1 = 0$,

(ii) $x^5 + 1 + (x^3 + 1)(x^2 - x + 1) = 0$,

(iii) $(x^2 + x + 1)^3 + x^6 - x^3 + 1 = 0$,

(iv) $(x^2 + x + 1)^4 + (x^2 - x + 1)^4 + x^8 + 1 = 0$.

3. Find a substitution of the form $x = my + n$ which will transform the equation $3x^4 + x^3 + 3x^2 + 31x + 10 = 0$ into a reciprocal equation.

Utilise this to solve the equation.

4. Find a substitution of the form $x = my$ which will transform the equation $x^4 + 5x^3 + 14x^2 + 20x + 16 = 0$ into a reciprocal equation.

Utilise this to solve the equation.

5. Diminish the roots of the equation $x^4 - 8x^3 + 20x^2 - 24x + 12 = 0$ by 1 and hence solve the given equation.

6. Increase the roots of the equation $x^4 + 7x^3 + 20x^2 + 27x + 15 = 0$ by 2 and hence solve the given equation.

7. If $\omega_1, \omega_2, \ldots, \omega_m$ be m distinct mth roots of unity, prove that

(i) $(a + b\omega_1)(a + b\omega_2)\ldots(a + b\omega_m) = a^m + (-1)^{m-1}b^m$,

(ii) $(a + b\omega_1)^m + (a + b\omega_2)^m + \cdots + (a + b\omega_m)^m = m(a^m + b^m)$.

[**Hint.** (i) Take $x = -\frac{a}{b}$ in the identity $x^m - 1 = (x - \omega_1)(x - \omega_2)\ldots(x - \omega_m)$.]

8. Solve the equation $x^7 - 1 = 0$. Deduce that $2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{8\pi}{7}$ are roots of the equation $t^3 + t^2 - 2t - 1 = 0$.

9. Solve the equation $x^{11} - 1 = 0$. Deduce that $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11} = \frac{1}{2^5}$.

10. If α be an imaginary root of $x^{11} - 1 = 0$, prove that

(i) $(\alpha + 1)(\alpha^2 + 1)\ldots(\alpha^{10} + 1) = 1$,

(ii) $(\alpha + 2)(\alpha^2 + 2)\ldots(\alpha^{10} + 2) = \frac{2^{11}+1}{3}$.

11. If α, β, γ be the roots of the equation $x^3 + 2x^2 + 1 = 0$, find the equation whose roots are

(i) $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}, \gamma + \frac{1}{\gamma}$; (ii) $\alpha + \beta\gamma, \beta + \gamma\alpha, \gamma + \alpha\beta$;

(iii) $2\alpha + \frac{1}{\alpha}, 2\beta + \frac{1}{\beta}, 2\gamma + \frac{1}{\gamma}$.

12. If α, β, γ be the roots of the equation $x^3 + 3x + 1 = 0$, find the equation whose roots are $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}, \frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}$.

13. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + 3x^2 + x + 1 = 0$, find the equation whose roots are

(i) $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}, \gamma + \frac{1}{\gamma}, \delta + \frac{1}{\delta}$; (ii) $\alpha + \frac{2}{\alpha}, \beta + \frac{2}{\beta}, \gamma + \frac{2}{\gamma}, \delta + \frac{2}{\delta}$;

(iii) $2\alpha + \frac{1}{\alpha}, 2\beta + \frac{1}{\beta}, 2\gamma + \frac{1}{\gamma}, 2\delta + \frac{1}{\delta}$.

14. If α be a special root of the equation $x^8 - 1 = 0$, prove that

(i) $(\alpha + 2)(\alpha^2 + 2)\ldots(\alpha^7 + 2) = \frac{2^8-1}{3}$,

(ii) $1 + 3\alpha + 5\alpha^2 + \cdots + 15\alpha^7 = \frac{16}{\alpha - 1}$.

15. If α be a special root of the equation $x^{12} - 1 = 0$, prove that $(\alpha + \alpha^{11})(\alpha^5 + \alpha^7) = -3$.

16. Find the special roots of the equation $x^9 - 1 = 0$. Deduce that $2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}$ are the roots of the equation $x^3 - 3x + 1 = 0$.

17. Find the special roots of the equation $x^{15} - 1 = 0$. Deduce that $2\cos\frac{2\pi}{15}, 2\cos\frac{4\pi}{15}, 2\cos\frac{8\pi}{15}, 2\cos\frac{16\pi}{15}$ are the roots of the equation $x^4 - x^3 - 4x^2 + 4x + 1 = 0$.

18. Find the special roots of the equation $x^{20} - 1 = 0$. Deduce that $\cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$ are the roots of the equation $16t^4 - 20t^2 + 5 = 0$.

19. Show that the special roots of the equation $x^{10} - 1 = 0$ are the non-real roots of the equation $x^5 + 1 = 0$.

20. If n be a prime number, prove that the special roots of the equation $x^{2n} - 1 = 0$ are the non-real roots of the equation $x^n + 1 = 0$.

21. If n be an odd positive integer, prove that the equations $x^{2n} - 1 = 0$ and $x^n - 1 = 0$ have the same number of special roots.

22. Find the number of special roots of the equation

(i) $x^{21} - 1 = 0$, (ii) $x^{32} - 1 = 0$, (iii) $x^{72} - 1 = 0$, (iv) $x^{90} - 1 = 0$.

5.10. Equations with binomial coefficients.

The general equation of degree n can be taken in the form

$$a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{2}a_2x^{n-2} + \cdots + a_n = 0,$$

where the coefficients of x^r is ${}^nC_r a_r$, the numerical component ${}^nC_r$ being the numerical coefficient of x^r in the expansion of $(1+x)^n$.

The equation having such a form is said to be an equation with *binomial coefficients*. Denoting such a polynomial of degree n by U_n, we have

$$U_1 = a_0x + a_1,$$
$$U_2 = a_0x^2 + 2a_1x + a_2,$$
$$U_3 = a_0x^3 + 3a_1x^2 + 3a_2x + a_3,$$
$$U_4 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4,$$
$$\cdots \quad \cdots \quad \cdots$$

It is easy to deduce that the derived polynomial of U_n is nU_{n-1}.

The main advantage of taking an equation with binomial coefficients is that it becomes very convenient to express the transformed equation when the roots are diminished by h.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$f(x) \equiv a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{2}a_2x^{n-2} + \cdots + a_n = 0,$$

then the equation whose roots are $\alpha_1 - h, \alpha_2 - h, \ldots, \alpha_n - h$ is $f(x+h) = 0$.

Now $f(x+h) = f(h) + xf'(h) + \frac{x^2}{2}f''(h) + \cdots + \frac{x^n}{n!}f^n(h)$.

If $f(x) = U_n(x)$ then
$$f'(x) = nU_{n-1}(x),$$
$$f''(x) = n(n-1)U_{n-2}(x),$$
$$\cdots$$
$$f^{n-1}(x) = n(n-1)\ldots 2U_1(x),$$
$$f^n(x) = n!a_0.$$

$$f(x+h) = a_0x^n + nU_1(h)x^{n-1} + \frac{n(n-1)}{1.2}U_2(h)x^{n-2} + \cdots + U_n(h)$$
$$= a_0x^n + n(a_0h + a_1)x^{n-1} + \frac{n(n-1)}{1.2}(a_0h^2 + 2a_2h + a_2)x^{n-2} + \cdots +$$
$$(a_0h^n + na_1h^{n-1} + \cdots + a_n)$$
$$= A_0x^n + nA_1x^{n-1} + \frac{n(n-1)}{1.2}A_2x^{n-2} + \cdots + A_n.$$

This shows that the transformed equation is also an equation with binomial coefficients.

5.11. The cubic equation.

The general form of a cubic equation with binomial coefficients is

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \quad \ldots \quad \text{(i)}$$

Let us apply the transformation $x = y + h$ in order that the transformed equation may want the second term.

The transformed equation is

$$a_0(y+h)^3 + 3a_1(y+h)^2 + 3a_2(y+h) + a_3 = 0$$

or, $a_0y^3 + 3(a_0h + a_1)y^2 + 3(a_0h^2 + 2a_1h + a_2)y + (a_0h^3 + 3a_1h^2 + 3a_2h + a_3) = 0$.

Since the equation wants the second term, $h = -\frac{a_1}{a_0}$ and the equation reduces to

$$y^3 + 3\frac{(a_0a_2 - a_1^2)}{a_0^2}y + \frac{(a_0^2a_3 - 3a_0a_1a_2 + a_1^3)}{a_0^3} = 0.$$

Using the standard symbols $H = a_0a_2 - a_1^2, G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$, the equation takes the form

$$y^3 + \frac{3H}{a_0^2}y + \frac{G}{a_0^3} = 0 \ldots \text{(ii)}$$

The roots of the equation are $\alpha + \frac{a_1}{a_0}, \beta + \frac{a_1}{a_0}, \gamma + \frac{a_1}{a_0}$ where α, β, γ are the roots of the cubic equation (i).

Since $\alpha + \beta + \gamma = -\frac{3a_1}{a_0}$, the roots of the equation (ii) are $\frac{1}{3}(2\alpha - \beta - \gamma), \frac{1}{3}(2\beta - \gamma - \alpha), \frac{1}{3}(2\gamma - \alpha - \beta)$.

Multiplying the roots of this equation by a_0, the transformed equation becomes $z^3 + 3Hz + G = 0$.

This is called the **standard form** of a cubic equation.

The roots of this equation are $a_0\alpha + a_1, a_0\beta + a_1, a_0\gamma + a_1$

i.e., $\frac{a_0}{3}(2\alpha - \beta - \gamma), \frac{a_0}{3}(2\beta - \gamma - \alpha), \frac{a_0}{3}(2\gamma - \alpha - \beta)$.

Note. $\Sigma(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha) = \frac{27H}{a_0^2}$ and

$(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta) = -\frac{27G}{a_0^3}$.

5.11.1. The equation whose roots are squares of the differences of the roots of a cubic equation.

Let α, β, γ be the roots of the cubic equation $x^3 + qx + r = 0$.

To find the equation whose roots are $(\beta - \gamma)^2, (\gamma - \alpha)^2, (\alpha - \beta)^2$.

Let $y = (\beta - \gamma)^2$.

Then $y = (\beta + \gamma)^2 - 4\beta\gamma = \alpha^2 + \frac{4r}{\alpha}$, since $\alpha + \beta + \gamma = 0, \quad \alpha\beta\gamma = -r$.

Therefore $\alpha^3 - \alpha\gamma + 4r = 0$.

Since $\alpha^3 + q\alpha + r = 0$, we have $(q+y)\alpha - 3r = 0$, or $\alpha = \frac{3r}{y+q}$.

Hence $(\frac{3r}{y+q})^3 + q(\frac{3r}{y+q}) + r = 0$

or, $(y+q)^3 + 3q(y+q)^2 + 27r^2 = 0$

or, $y^3 + 6qy^2 + 9q^2y + 27r^2 + 4q^3 = 0$. ⋯ (A)

This is the required equation.

If it is proposed to form an equation whose roots are squares of the differences of the roots of the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \quad \cdots \quad \text{(i)}$$

we first remove the second term. The transformed equation is

$$y^3 + \frac{3H}{a_0^2}y + \frac{G}{a_0^3} = 0 \quad \cdots \quad \text{(ii)}$$

and its roots are $\alpha + \frac{a_1}{a_0}, \beta + \frac{a_1}{a_0}, \gamma + \frac{a_1}{a_0}$ where α, β, γ are the roots of the equation (i).

Let $\alpha' = \alpha + \frac{a_1}{a_0}, \beta' = \beta + \frac{a_1}{a_0}, \gamma' = \gamma + \frac{a_1}{a_0}$.

Then $\beta' - \gamma' = \beta - \gamma, \gamma' - \alpha' = \gamma - \alpha, \alpha' - \beta' = \alpha - \beta$.

Therefore the equation whose roots are squares of the differences of the roots of the cubic equation (i) is same as the equation whose roots are $(\beta' - \gamma')^2, (\gamma' - \alpha')^2, (\gamma' - \beta')^2$, and the equation can be obtained by putting $q = \frac{3H}{a_0^2}, r = \frac{G}{a_0^3}$ in (A).

Therefore the required equation is

$$x^3 + \frac{18H}{a_0^2}x^2 + \frac{81H^2}{a_0^4}x + \frac{27}{a_0^6}(G^2 + 4H^3) = 0. \quad \cdots \quad \text{(B)}$$

Note 1. It follows that $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = -\frac{27}{a_0^6}(G^2 + 4H^3)$.

Note 2. Nature of the roots.

Assuming that the coefficients are all real, we discuss the nature of the roots of the equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$.

Case I. $G^2 + 4H^3 > 0$.

In this case $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 < 0$. The cubic has two imaginary roots, because otherwise, if all the roots be real then each of $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$ is non-negative and therefore their product cannot be negative.

Case II. $G^2 + 4H^3 < 0$ and $H < 0$.

In this case the signs of the coefficients of the equation (B) are alternately positive and negative and therefore, by Descartes' rule of signs, the equation (B) has no negative root and consequently, the given equation has all its roots real. Because otherwise, if $\lambda + \mu i$ be a root of the given cubic then $\lambda - \mu i$ will be another root and the square of their difference

which is a root of the equation (B) is obviously a negative real number, a contradiction.

Case III. $G^2 + 4H^3 = 0$.

In this case one of $(\alpha - \beta), (\beta - \gamma), (\gamma - \alpha)$ is zero and this proves the existence of a multiple root of the given cubic.

Case IV. $G^2 + 4H^3 = 0$ and $H = 0$.

In this case the equation (B) reduces to $x^3 = 0$ and this proves that $(\alpha - \beta)^2 = 0, (\beta - \gamma)^2 = 0, (\gamma - \alpha)^2 = 0$. Therefore the given cubic has three equal roots.

Worked Examples.

1. Find the equation whose roots are squares of the differences of the roots of the equation $x^3 + 9x^2 + 24x + 20 = 0$. What conclusion do you draw about the nature of the roots of the given equation?

Let α, β, γ be the roots of the given equation.

Let us apply the transformation $x = y + h$ in order to remove the second term.

The equation transforms to

$(y+h)^3 + 9(y+h)^2 + 24(y+h) + 20 = 0$

or, $y^3 + (3h+9)y^2 + (3h^2 + 18h + 24)y + (h^3 + 9h^2 + 24h + 20) = 0$.

Since the second term of this equation is to be absent, $h = -3$.

The transformed equation is $y^3 - 3y + 2 = 0$.

The roots of the transformed equation are $\alpha + 3, \beta + 3, \gamma + 3$.

Let $\alpha' = \alpha + 3, \beta' = \beta + 3, \gamma' = \gamma + 3$.

Therefore the equation whose roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$ is same as the equation whose roots are $(\alpha' - \beta')^2, (\beta' - \gamma')^2, (\gamma' - \alpha')^2$.

Let $z = (\alpha' - \beta')^2$. Then $z = (\alpha' + \beta')^2 - 4\alpha'\beta' = \gamma'^2 + \frac{8}{\gamma'}$, since $\alpha' + \beta' + \gamma' = 0$, $\alpha'\beta'\gamma' = -2$.

Therefore $\gamma'^3 - \gamma' z + 8 = 0$.

Since $\gamma'^3 - 3\gamma' + 2 = 0$, we have $\gamma'(z - 3) = 6$, or $\gamma' = \frac{6}{z-3}$.

Hence $(\frac{6}{z-3})^3 - 3(\frac{6}{z-3}) + 2 = 0$

or, $2(z-3)^3 - 18(z-3)^2 + 216 = 0$

or, $z^3 - 18z^2 + 81z = 0$.

This is the required equation.

One root of the transformed equation, say $(\alpha - \beta)^2$ is zero. That is, two roots of the given equation are equal.

2. Find the equation whose roots are squares of the differences of the roots of the equation $x^3 + x + 2 = 0$ and deduce from the resulting equation the nature of the roots of the given cubic.

Let α, β, γ be the roots of the given equation.
Let $y = (\beta - \gamma)^2$.
Then $y = (\beta + \gamma)^2 - 4\beta\gamma = \alpha^2 + \frac{8}{\alpha}$, since $\alpha + \beta + \gamma = 0$, $\alpha\beta\gamma = -2$
Therefore $\alpha^3 - \alpha\gamma + 8 = 0$.
Since $\alpha^3 + \alpha + 2 = 0$, we have $\alpha(y+1) = 6$, or $\alpha = \frac{6}{y+1}$.

Hence $(\frac{6}{y+1})^3 + \frac{6}{y+1} + 2 = 0$
or, $(y+1)^3 + 3(y+1)^2 + 108 = 0$
or, $y^3 + 6y^2 + 9y + 112 = 0$.

This is the required equation.

By Descartes' rule of signs, this equation has at least one negative root. Therefore the given cubic must have a pair of imaginary roots.

5.11.2. The general solution of a cubic. Cardan's method.

Let the cubic equation be $ax^3 + 3bx^2 + 3cx + d = 0$ ... (i)
This can be put in the standard form $z^3 + 3Hz + G = 0$,
where $z = ax + b$, $H = ac - b^2$, $G = a^2d - 3abc + 2b^3$.

To solve the equation, let us assume $z = u + v$.
Then $z^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvz$
or, $z^3 - 3uvz - (u^3 + v^3) = 0$.
Comparing this with $z^3 + 3Hz + G = 0$, we have
$uv = -H \quad u^3 + v^3 = -G$.

Therefore $u^3 = \frac{1}{2}(-G + \sqrt{G^2 + 4H^3})$, $v^3 = \frac{1}{2}(-G - \sqrt{G^2 + 4H^3})$.

If p denotes any one of the three values of $\{\frac{1}{2}(-G + \sqrt{G^2 + 4H^3})\}^{1/3}$, then the three values of u are $p, \omega p, \omega^2 p$ where ω is an imaginary cube root of unity.

And since $uv = -H$, the three coresponding values of v are $\frac{-H}{p}, \frac{-\omega^2 H}{p}, \frac{-\omega H}{p}$.

Hence the values of z are $p - \frac{H}{p}, \omega p - \frac{\omega^2 H}{p}, \omega^2 p - \frac{\omega H}{p}$ and the three values of x are $\frac{1}{a}(p - \frac{H}{p} - b), \frac{1}{a}(\omega p - \frac{\omega^2 H}{p} - b), \frac{1}{a}(\omega^2 p - \frac{\omega H}{p} - b)$.

This gives the complete solution of the equation (i).

The method of solution is called the Cardan's method of solution although the method owes its origin to Tartaglia.

Note. When $G^2 + 4H^3 < 0$, the roots of the cubic equation are all real but Cardan's solution give them in imaginary form.

In this case we use De Moivre's theorem to obtain the real roots in the following manner.

Let $G^2 + 4H^3 = -k^2$.
Then $u^3 = \frac{1}{2}(-G + ik)$, $v^3 = \frac{1}{2}(-G - ik)$.
Let $\frac{-G}{2} = r\cos\theta$, $\frac{k}{2} = r\sin\theta$, where $-\pi < \theta \le \pi$.
Then $u^3 = r(\cos\theta + i\sin\theta)$ and $r^2 = -H^3$.
Therefore the three values of u are $\sqrt[3]{r}(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3})$, $\sqrt[3]{r}(\cos\frac{2\pi+\theta}{3} + i\sin\frac{2\pi+\theta}{3})$, $\sqrt[3]{r}(\cos\frac{4\pi+\theta}{3} + i\sin\frac{4\pi+\theta}{3})$.
Also since $uv = -H$, the corresponding values of v are $\sqrt[3]{r}(\cos\frac{\theta}{3} - i\sin\frac{\theta}{3})$, $\sqrt[3]{r}(\cos\frac{2\pi+\theta}{3} - i\sin\frac{2\pi+\theta}{3})$, $\sqrt[3]{r}(\cos\frac{4\pi+\theta}{3} - i\sin\frac{4\pi+\theta}{3})$.
Hence the values of z are $2\sqrt[3]{r}\cos\frac{\theta}{3}, 2\sqrt[3]{r}\cos\frac{2\pi+\theta}{3}, 2\sqrt[3]{r}\cos\frac{4\pi+\theta}{3}$,
i.e., $2\sqrt{-H}\cos\frac{\theta}{3}, 2\sqrt{-H}\cos\frac{2\pi+\theta}{3}, 2\sqrt{-H}\cos\frac{4\pi+\theta}{3}$.

Worked Examples.

1. Solve the equation $x^3 - 18x - 35 = 0$.

Let $x = u + v$.
Then $x^3 = u^3 + v^3 + 3uvx$
or, $x^3 - 3uvx - (u^3 + v^3) = 0$.

Comparing with the given cubic, we have $uv = 6$ and $u^3 + v^3 = 35$.

Therefore $u^3 = \frac{1}{2}(35 + \sqrt{35^2 - 864}) = 27$ and

$$v^3 = \frac{1}{2}(35 - \sqrt{35^2 - 864}) = 8.$$

The three values of u are $3, 3\omega, 3\omega^2$ and the three values of v are $2, 2\omega, 2\omega^2$. Since $uv = 6$, we have $u + v = 3 + 2, 3\omega + 2\omega^2, 3\omega^2 + \omega$.

Hence the roots of the given equation are $5, \frac{-5+\sqrt{3}i}{2}, \frac{-5-\sqrt{3}i}{2}$.

2. Solve the equation $x^3 - 15x^2 - 33x + 847 = 0$.

Let us apply the transformation $x = y + h$ in order to remove the second term.

The transformed equation is
$(y+h)^3 - 15(y+h)^2 - 33(y+h) + 847 = 0$
or, $y^3 + (3h - 15)y^2 + (3h^2 - 30h - 33)y + (h^3 - 15h^2 - 33h + 847) = 0$.
So $h = 5$ and the equation reduces to $y^3 - 108y + 432 = 0$ ... (i)

Let $y = u + v$.
Then $y^3 = u^3 + v^3 + 3uvy$
or, $y^3 - 3uvy - (u^3 + v^3) = 0$.

Comparing with the equation (i), we have $uv = 36$ and $u^3 + v^3 = -432$. Therefore $u^3 = v^3 = -216$.

The three values of u are $-6, -6\omega, -6\omega^2$.
Since $uv = 36$, the corresponding values of v are $-6, -6\omega^2, -6\omega$.
Then $y = -12, 6, 6$ and the roots of the given equation are $-7, 11, 11$.

3. Solve the equation $x^3 - 3x - 1 = 0$.

Let $x = u + v$.
Then $x^3 = u^3 + v^3 + 3uvx$
or, $x^3 - 3uvx - (u^3 + v^3) = 0$.

Comparing with the given cubic, we have $uv = 1$ and $u^3 + v^3 = 1$.

Therefore $u^3 = \frac{1}{2}(1 + \sqrt{3}i), v^3 = \frac{1}{2}(1 - \sqrt{3}i)$.

or, $u = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^{\frac{1}{3}}, v = (\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})^{\frac{1}{3}}$.

The three values of u are $\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}, \cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}, \cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}$
and the three values of v are $\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}, \cos\frac{7\pi}{9} - i\sin\frac{7\pi}{9}, \cos\frac{13\pi}{9} - i\sin\frac{13\pi}{9}$.

Since $uv = 1$,

$u = \cos\frac{\pi}{9} + i\sin\frac{\pi}{9}$ corresponds to $v = \cos\frac{\pi}{9} - i\sin\frac{\pi}{9}$;

$u = \cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}$ corresponds to $v = \cos\frac{7\pi}{9} - i\sin\frac{7\pi}{9}$;

$u = \cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}$ corresponds to $v = \cos\frac{13\pi}{9} - i\,;\sin\frac{13\pi}{9}$.

Taking $x = u + v$, the roots of the given equation are $2\cos\frac{\pi}{9}, 2\cos\frac{7\pi}{9}, 2\cos\frac{13\pi}{9}$.

Note. The roots of this cubic equation are all real.

Exercises 5F

1. Find the equation whose roots are squares of the differences of the roots of the equation

(i) $x^3 + 3x + 1 = 0$,

(ii) $x^3 + 6x^2 + 9x + 4 = 0$,

(iii) $x^3 + 3x^2 - 24x + 28 = 0$.

2. Determine the nature of the roots of the equation in Ex.1.

3. If α, β, γ be the roots of the equation $x^3 - 3qx + r = 0$, show that

(i) $(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) = \pm\sqrt{27(4q^3 - r^2)}$,

(ii) $\Sigma(\alpha - \beta)(\beta - \gamma) = -9q$.

4. If α, β, γ be the roots of the equation $x^3 - 3qx + r = 0$, show that

(i) $(\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha) + (\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2) = 3r$,

(ii) $(\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha)(\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2) = -27q^3 + 9r^2$,

(iii) $\alpha^3\beta + \beta^3\gamma + \gamma^3\alpha = \alpha\beta^3 + \beta\gamma^3 + \gamma\alpha^3 = -9q^2$.

5. If α, β, γ be the roots of the equation $x^3 - 9x + 9 = 0$, prove that
$(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) = \pm 27$.

6. If α, β, γ be the roots of the equation $x^3 - 3x - 1 = 0$, show that

(i) the equation whose roots are $(\alpha-\beta)(\alpha-\gamma), (\beta-\gamma)(\beta-\alpha), (\gamma-\alpha)(\gamma-\beta)$ is $x^3 - 9x^2 + 81 = 0$;

(ii) the equation whose roots are $\alpha - \beta, \beta - \gamma, \gamma - \alpha$ is $x^3 - 9x \pm 9 = 0$.

7. If $a + b + c = 0, a^2 + b^2 + c^2 = 42, a^3 + b^3 + c^3 = 105$, show that
$(a - b)(b - c)(c - a) = \pm 63$.

8. If α, β, γ be the roots of the equation $x^3 - qx + r = 0$ find the relation between q and r so that $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$ may be in

(i) arithmetic progression, (ii) geometric progression.

9. If α, β, γ be the roots of the equation $x^3 - 3x + 1 = 0$, prove that

(i) $\begin{vmatrix} \alpha^3 & \alpha^2 & 1 \\ \beta^3 & \beta^3 & 1 \\ \gamma^3 & \gamma^2 & 1 \end{vmatrix} = \pm 27$, (ii) $\begin{vmatrix} \alpha^4 & \alpha & 1 \\ \beta^4 & \beta & 1 \\ \gamma^4 & \gamma & 1 \end{vmatrix} = \pm 27$.

10. Solve by Cardan's method

(i) $x^3 - 27x - 54 = 0$,

(ii) $x^3 - 9x + 28 = 0$,

(iii) $x^3 - 12x + 8 = 0$,

(iv) $x^3 - 3x - 2\cos A = 0\ (-\pi < A \leq \pi)$,

(v) $x^3 - 6x + 4 = 0$,

(vi) $9x^3 - 9x - 4 = 0$,

(vii) $2x^3 - 3x + 1 = 0$,

(viii) $x^3 + 9x^2 + 15x - 25 = 0$,

(ix) $x^3 - 6x^2 - 6x - 7 = 0$,

(x) $x^3 + 3x^2 - 3 = 0$.

11. Prove that the relation among the coefficients a_0, a_2, a_3 so that the equation $a_0x^3 + a_2x + a_3 = 0$ may be put in the form $x^4 = (x^2 + px + q)^2$ is

$$a_2{}^3 + 8a_0a_3{}^2 = 0.$$

Utilise this to solve the equation

(i) $125x^3 - 10x - 1 = 0$,

(ii) $64x^3 - 72x + 27 = 0$,

(iii) $27x^3 - 6x + 1 = 0$,

(iv) $8x^3 - 36x - 27 = 0$.

[**Hint.** (i) $p = \frac{a_2{}^2}{2a_0a_3} = -\frac{2}{5}, q = \frac{a_2}{a_0} = -\frac{2}{25}$.]

12. Use the identity

$(x+p+q)(x+\omega p+\omega^2 q)(x+\omega^2 p+\omega q) = x^3+p^3+q^3-3pqx$ to solve the equation

(i) $x^3 - 12x + 16 = 0$, (ii) $x^2 - 12x + 65 = 0$.

5.12. The biquadratic equation.

The general form of a biquadratic equation with binomial coefficients is

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0. \quad \cdots \quad \text{(i)}$$

Let us apply the transformation $x = y + h$ in order that the transformed equation may want the second term.

The transformed equation is

$$a_0(y+h)^4 + 4a_1(y+h)^3 + 6a_2(y+h)^2 + 4a_3(y+h) + a_4 = 0$$

or, $a_0y^4 + 4(a_0h + a_1)y^3 + 6(a_0h^2 + 2a_1h + a_2)y^2 + 4(a_0h^3 + 3a_1h^2 + 3a_2h + a_3)y + (a_0h^4 + 4a_1h^3 + 6a_2h^2 + 4a_3h + a_4) = 0.$

Since the transformed equation wants the second term $h = -\frac{a_1}{a_0}$ and the equation reduces to

$$a_0y^4 + \tfrac{6}{a_0}(a_0a_2 - a_1{}^2)y^2 + \tfrac{4}{a_0{}^2}(a_0{}^2a_3 - 3a_0a_1a_2 + 2a_1{}^3)y + \tfrac{1}{a_0{}^3}(a_0{}^3a_4 - 4a_0{}^2a_1a_3 + 6a_0a_1{}^2a_2 - 3a_1{}^4) = 0.$$

Using the standard symbols $H = a_0a_2 - a_1{}^2, G = a_0{}^2a_3 - 3a_0a_1a_2 + 2a_1{}^3$ and $I = a_0a_4 - 4a_1a_3 + 3a_2{}^2$, the equation takes the form

$$a_0y^4 + \tfrac{6H}{a_0}y^2 + \tfrac{4G}{a_0{}^2}y + \tfrac{1}{a_0{}^3}(a_0{}^2I - 3H^2) = 0.$$

The roots of the equation are $\alpha + \frac{a_1}{a_0}, \beta + \frac{a_1}{a_0}, \gamma + \frac{a_1}{a_0}, \delta + \frac{a_1}{a_0}$ where $\alpha, \beta, \gamma, \delta$ are the roots of the equation (i).

Since $\alpha + \beta + \gamma + \delta = -\frac{4a_1}{a_0}$, the roots are $\frac{1}{4}(3\alpha - \beta - \gamma - \delta), \frac{1}{4}(3\beta - \gamma - \delta - \alpha), \frac{1}{4}(3\gamma - \delta - \alpha - \beta), \frac{1}{4}(3\delta - \alpha - \beta - \gamma)$.

Multiplying the roots by a_0, the transformed equation is

$$z^4 + 6Hz^2 + 4Gz + (a_0{}^2I - 3H^2) = 0.$$

This is called the **standard form** of a biquadratic equation.

The roots of this equation are $\frac{1}{4}a_0(3\alpha - \beta - \gamma - \delta), \frac{1}{4}a_0(3\beta - \gamma - \delta - \alpha), \frac{1}{4}a_0(3\gamma - \delta - \alpha - \beta), \frac{1}{4}a_0(3\delta - \alpha - \beta - \gamma)$.

Worked Example.

1. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, find the value of

(i) $(\alpha + \beta - \gamma - \delta)(\beta + \gamma - \alpha - \delta)(\gamma + \alpha - \beta - \delta)$,

(ii) $(3\alpha - \beta - \gamma - \delta)(3\beta - \gamma - \alpha - \delta)(3\gamma - \alpha - \beta - \delta)(3\delta - \alpha - \beta - \gamma)$.

Let us apply the transformation $x = y + h$ in order to remove the second term. Then

$$a_0(y+h)^4 + 4a_1(y+h)^3 + 6a_2(y+h)^2 + 4a_3(y+h) + a_4 = 0$$

or, $a_0y^4 + 4(a_0h + a_1)y^3 + 6(a_0h^2 + 2a_1h + a_2)y^2 + 4(a_0h^3 + 3a_1h^2 + 3a_2h + a_3)y + (a_0h^4 + 4a_1h^3 + 6a_2h^2 + 4a_3h + a_4) = 0.$

Therefore $h = -\frac{a_1}{a_0}$ and the equation reduces to

$$a_0y^4 + 6(\tfrac{a_0a_2 - a_1{}^2}{a_0})y^2 + 4(\tfrac{a_0{}^2a_3 - 3a_0a_1a_2 + 2a_1{}^3}{a_0{}^2})y + (\tfrac{a_4a_0{}^3 - 4a_0{}^2a_1a_3 + 6a_0a_1{}^2a_2 - 3a_1{}^4}{a_0{}^4}) = 0.$$

Let $\alpha', \beta', \gamma', \delta'$ be the roots of the transformed equation.

Then $\alpha = \alpha' - \frac{a_1}{a_0}, \beta = \beta' - \frac{a_1}{a_0}, \gamma = \gamma' - \frac{a_1}{a_0}, \delta = \delta' - \frac{a_1}{a_0}$.

(i) $(\alpha + \beta - \gamma - \delta)(\beta + \gamma - \alpha - \delta)(\gamma + \alpha - \beta - \delta)$

$= (\alpha' + \beta' - \gamma' - \delta')(\beta' + \gamma' - \alpha' - \delta')(\gamma' + \alpha' - \beta' - \delta')$

$= -8(\gamma' + \delta')(\alpha' + \delta')(\beta' + \delta')$, since $\alpha' + \beta' + \gamma' + \delta' = 0$

$= -8[\delta'^3 + \delta'^2(\alpha' + \beta' + \gamma') + \delta'(\alpha'\beta' + \beta'\gamma' + \gamma'\alpha') + \alpha'\beta'\gamma']$

$= -8\Sigma\alpha'\beta'\gamma'$, since $\alpha' + \beta' + \gamma' = -\delta'$

$= 32(\frac{a_0{}^2a_3 - 3a_0a_1a_2 + 2a_1{}^3}{a_0{}^3}) = \frac{32G}{a_0{}^3}$.

(ii) $(3\alpha - \beta - \gamma - \delta)(3\beta - \gamma - \delta - \alpha)(3\gamma - \delta - \alpha - \beta)(3\delta - \alpha - \beta - \gamma)$

$= (3\alpha' - \beta' - \gamma' - \delta')(3\beta' - \gamma' - \delta' - \alpha')(3\gamma' - \delta' - \alpha' - \beta')(3\delta' - \alpha' - \beta' - \gamma')$

$= 4\alpha'.4\beta'.4\gamma'.4\delta'$, since $\alpha' + \beta' + \gamma' + \delta' = 0$

$= 256(\frac{a_4a_0{}^3 - 4a_0{}^2a_1a_3 + 6a_0a_1{}^2a_2 - 3a_1{}^4}{a_0{}^4}) = 256(\frac{a_0{}^2I - 3H^2}{a_0{}^4})$.

5.12.1. Ferrai's solution of a biquadratic equation.

Ferrari's method reduces the problem of solving a biquadratic equation to that of solving two quadratic equations. This is done by expressing the biquadratic as the difference of two perfect suqares.

Let the equation be $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$.
Multiplying by a, $a^2x^4 + 4abx^3 + 6acx^2 + 4adx + ae = 0$ ... (i)

Let the left hand expression be expressed as the difference of two squares in the form
$$(ax^2 + 2bx + \lambda)^2 - (mx + n)^2.$$
Comparing with the left hand expression of (i) we have
$$6ac = 4b^2 + 2\lambda a - m^2,$$
$$4ad = 4b\lambda - 2mn,$$
$$ae = \lambda^2 - n^2.$$

Eliminating m, n from these we have
$$4(b\lambda - ad)^2 = (2\lambda a + 4b^2 - 6ac)(\lambda^2 - ae).$$

This is a cubic equation in λ, giving at least one real root λ_1.

Corresponding to $\lambda = \lambda_1$, we have the values of m^2 and n^2 and morever the relation $mn = 2b\lambda_1 - 2ad$ determines only one value of n corresponding to one value of m.

Thus the given equation is now put in the form
$(ax^2 + 2bx + \lambda_1)^2 - (m_1x + n_1)^2 = 0$, where m_1, n_1 are the values of m, n corresponding to λ_1.

The roots of the quadratic equations $ax^2 + 2bx + \lambda_1 \pm (m_1x + n_1) = 0$ give the solution of the given biquadratic equation.

Worked Examples.

1. Solve by Ferrari's method
$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$$

The equation may be written as
$(x^2 - 5x + \lambda)^2 - (mx + n)^2 = 0$, where λ, m, n are constants.

Equating coefficients of like powers of x
$$35 = 25 + 2\lambda - m^2 \quad \text{or, } m^2 = 2\lambda - 10;$$
$$-50 = -10\lambda - 2mn \quad \text{or, } mn = -5\lambda + 25;$$
$$24 = \lambda^2 - n^2 \quad \text{or, } n^2 = \lambda^2 - 24.$$

Eliminating m, n we have
$$(\lambda^2 - 24)(2\lambda - 10) - (5\lambda - 25)^2 = 0$$
$$\text{or, } (\lambda - 5)[2\lambda^2 - 48 - 25\lambda + 125] = 0$$

$$\text{or, } (\lambda - 5)[2\lambda^2 - 25\lambda + 77] = 0.$$
Therefore $\lambda = 5, 7, \frac{11}{2}$.

Taking $\lambda = 5$, we have $m = 0, n = \pm 1$.
The equation takes the form
$$(x^2 - 5x + 5)^2 - 1 = 0$$
$$\text{or, } (x^2 - 5x + 6)(x^2 - 5x + 4) = 0$$
$$\text{or, } (x - 2)(x - 3)(x - 1)(x - 4) = 0.$$
Therefore $x = 2, 3, 1, 4$.

Hence the roots of the equation are $1, 2, 3, 4$.

2. If $f(x) = x^4 + 6x^2 + 14x^2 + 22x + 5$ find α, β, λ so that $f(x)$ may be expressed in the form $(x^2 + 3x + \lambda)^2 - (\alpha x + \beta)^2$. Hence solve the equation $f(x) = 0$.

$$x^4 + 6x^2 + 14x^2 + 22x + 5 = (x^2 + 3x + \lambda)^2 - (\alpha x + \beta)^2.$$

Equating coefficients of like powers of x, we have
$$14 = 9 + 2\lambda - \alpha^2, \text{ or } \alpha^2 = 2\lambda - 5;$$
$$22 = 6\lambda - 2\alpha\beta, \text{ or } \alpha\beta = 3\lambda - 11;$$
$$5 = \lambda^2 - \beta^2, \text{ or } \beta^2 = \lambda^2 - 5.$$

Eliminating α, β we have
$$(\lambda^2 - 5)(2\lambda - 5) - (3\lambda - 11)^2 = 0$$
$$\text{or, } 2\lambda^3 - 14\lambda^2 + 56\lambda - 96 = 0$$
$$\text{or, } (\lambda - 3)(2\lambda^2 - 8\lambda + 32) = 0.$$
$$\text{Therefore } \lambda = 3, 2 \pm 2\sqrt{3}i.$$

Taking $\lambda = 3$, we have $\alpha = \pm 1, \beta = \pm 2$ and $\alpha\beta = -2$.

Therefore α and β are of opposite signs.

The equation can be expressed as
$$(x^2 + 3x + 3)^2 - (x - 2)^2 = 0$$
$$\text{or, } (x^2 + 4x + 1)(x^2 + 2x + 5) = 0.$$

Hence the roots of the equation are $-2 \pm \sqrt{3}, -1 \pm 2i$.

5.12.2. Euler's method of solution of a biquadratic equation.

Let the equation be $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ ... (i)

By the transformation $y = x + \frac{a_1}{a_0}$ the equation reduces to
$$a_0y^4 + \frac{6H}{a_0}y^2 + \frac{4G}{a_0^2}y + \frac{1}{a_0^3}(a_0^2I - 3H^2) = 0,$$
where $H = a_0a_2 - a_1^2, G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$ and $I = a_0a_4 - 4a_1a_3 + 3a_2^2$.

Again by the transformation $z = a_0 y$ the equation takes the standard form

$$z^4 + 6Hz^2 + 4Gz + (a_0{}^2 I - 3H^2) = 0 \quad \cdots \quad \text{(ii)}$$

To solve the equation, let $z = p^{1/2} + q^{1/2} + r^{1/2}$. Then

$\{z^2 - (p+q+r)\}^2 = 4\{pq+qr+rp+2p^{1/2}q^{1/2}r^{1/2}(p^{1/2}+q^{1/2}+r^{1/2})\}$

or, $z^4 - 2(p+q+r)z^2 - 8p^{\frac{1}{2}}q^{\frac{1}{2}}r^{\frac{1}{2}}z + (p+q+r)^2 - 4(pq+qr+rp) = 0.$

Comparing with (ii) we have

$$p+q+r = -3H,$$
$$p^{1/2}q^{1/2}r^{1/2} = -\frac{G}{2},$$
$$(p+q+r)^2 - 4(pq+qr+rp) = a_0{}^2 I - 3H^2.$$

Therefore $pq+qr+rp = 3H^2 - \frac{a_0{}^2 I}{4}$.

Hence p, q, r, are the roots of the cubic equation

$$t^3 + 3Ht^2 + (3H^2 - \tfrac{a_0{}^2 I}{4})t - \tfrac{G^2}{4} = 0. \quad \cdots \quad \text{(iii)}$$

This is called the *Euler's cubic.*

Corresponding to a root p there are two values of $p^{1/2}$. Considering two values of each of $p^{1/2}, q^{1/2}, r^{1/2}$ there will be eight values of z but they are restricted to satisfy the relation $p^{1/2}q^{1/2}r^{1/2} = -\frac{G}{2}$.

Taking into account this limitation, there will be only four values of z, the possible combinations being determined by

$\sqrt{p}\sqrt{q}\sqrt{r} = \sqrt{p}(-\sqrt{q})(-\sqrt{r}) = (-\sqrt{p})(-\sqrt{q})\sqrt{r} = (-\sqrt{p})\sqrt{q}(-\sqrt{r}) = -\frac{G}{2}$ if $G < 0$,

or by $(-\sqrt{p})(-\sqrt{q})(-\sqrt{r}) = (-\sqrt{p})\sqrt{q}\sqrt{r} = \sqrt{p}(-\sqrt{q})\sqrt{r} = \sqrt{p}\sqrt{q}(-\sqrt{r}) = -\frac{G}{2}$ if $G > 0$,

where $\sqrt{p}$ denotes the (positive) principal square root of p.

The Euler's cubic (iii) can be expressed as

$$(t+H)^3 - \tfrac{a_0{}^2 I}{4}t - \tfrac{G^2+4H^3}{4} = 0.$$

Using the standard symbol

$$J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3{}^2 - a_1{}^2a_4 - a_2{}^3$$
$$= \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix},$$

we have $G^2 + 4H^3 = a_0{}^2(HI - a_0 J)$.

The equation takes the form

$$(t+H)^3 - \tfrac{a_0{}^2 I}{4}(t+H) + \tfrac{a_0{}^3 J}{4} = 0.$$

Taking $t + H = a_0\theta$, the equation reduces to $4\theta^3 - I\theta + J = 0$.

This is called the *reducing cubic* of the biquadratic equation.

Let the roots of the reducing cubic be $\theta_1, \theta_2, \theta_3$. Then

$$\begin{aligned} a_0\theta_1 &= t + H \\ &= p + a_0a_2 - a_1{}^2. \end{aligned}$$

$$\begin{aligned} \text{Therefore } p &= a_1{}^2 - a_0a_2 + a_0\theta_1. \\ \text{Similarly } q &= a_1{}^2 - a_0a_2 + a_0\theta_2, \\ r &= a_1{}^2 - a_0a_2 + a_0\theta_3. \end{aligned}$$

$$\begin{aligned} \text{Hence } p - q &= a_0(\theta_1 - \theta_2), \\ q - r &= a_0(\theta_2 - \theta_3), \\ r - p &= a_0(\theta_3 - \theta_1). \end{aligned}$$

Relation between the roots of the biquadratic and the roots of the Euler's cubic.

Let the roots of the biquadratic equation (i) be $\alpha, \beta, \gamma d$.

Then four values of z are $a_0\alpha + a_1, a_0\beta + a_1, a_0\gamma + a_1, a_0\delta + a_1$.

Let us take

$$a_0\alpha + a_1 = \sqrt{p} - \sqrt{q} - \sqrt{r},$$
$$a_0\beta + a_1 = \sqrt{p} + \sqrt{q} - \sqrt{r},$$
$$a_0\gamma + a_1 = \sqrt{p} - \sqrt{q} + \sqrt{r},$$
$$a_0\delta + a_1 = \sqrt{p} + \sqrt{q} + \sqrt{r}.$$

Then $a_0{}^2(\beta + \gamma - \alpha - \delta)^2 = 16p,$

$$a_0{}^2(\gamma + \alpha - \beta - \delta)^2 = 16q,$$
$$a_0{}^2(\alpha + \beta - \gamma - \delta)^2 = 16r.$$

Relation between the roots of the biquadratic and the roots of the reducing cubic.

$$\begin{aligned} a_0(\theta_1-\theta_2) &= p-q \\ &= -\tfrac{1}{4}a_0{}^2(\alpha-\beta)(\gamma-\delta), \\ a_0(\theta_2-\theta_3) &= -\tfrac{1}{4}a_0{}^2(\beta-\gamma)(\alpha-\delta), \\ a_0(\theta_3-\theta_1) &= -\tfrac{1}{4}a_0{}^2(\gamma-\alpha)(\beta-\delta). \end{aligned}$$

Therefore $a_0(\theta_3-\theta_1)-a_0(\theta_1-\theta_2)$

$$= -\tfrac{1}{4}a_0{}^2\{(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)\}.$$

Since $\theta_1+\theta_2+\theta_3=0$,

$$12\theta_1 = a_0\{(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)\},$$
$$12\theta_2 = a_0\{(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta)\},$$
$$12\theta_3 = a_0\{(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta)\}.$$

Worked Examples.

1. Solve the equation $x^4-6x^2+16x-15=0$ by Euler's method.

Let $x=p^{1/2}+q^{1/2}+r^{1/2}$. Then

$x^4-2(p+q+r)x^2-8p^{1/2}q^{1/2}r^{1/2}x+(p+q+r)^2-4(pq+qr+rp)=0.$

Comparing the coefficients, we have

$$p+q+r=3,$$
$$p^{1/2}q^{1/2}r^{1/2}=-2,$$
$$(p+q+r)^2-4(pq+qr+rp)=-15.$$

Therefore $pq+qr+rp=6$.

Hence p,q,r are the roots of the cubic $t^3-3t^2+6t-4=0$.

The roots of the equation are $1+\sqrt{3}i, 1-\sqrt{3}i, 1$.

Let $p=1+\sqrt{3}i, q=1-\sqrt{3}i, r=1$.

Then $p^{1/2}=\pm\frac{\sqrt{3}+i}{\sqrt{2}}, q^{1/2}=\pm\frac{\sqrt{3}-i}{\sqrt{2}}, r^{1/2}=\pm1$.

But $p^{1/2}.q^{1/2}.r^{1/2}=-2$.

Hence the roots are $\frac{\sqrt{3}+i}{\sqrt{2}}+\frac{\sqrt{3}-i}{\sqrt{2}}-1, \frac{\sqrt{3}+i}{\sqrt{2}}-\frac{\sqrt{3}-i}{\sqrt{2}}+1, -\frac{\sqrt{3}+i}{\sqrt{2}}+\frac{\sqrt{3}-i}{\sqrt{2}}+1$ and $-\frac{\sqrt{3}+i}{\sqrt{2}}-\frac{\sqrt{3}-i}{\sqrt{2}}-1$.

i.e., $\sqrt{6}-1, \sqrt{2}i+1, -\sqrt{2}i+1$ and $-\sqrt{6}-1$.

2. Solve the equation $x^4-3x^2-6x-2=0$ by Euler's method.

Let $x=p^{1/2}+q^{1/2}+r^{1/2}$. Then

$x^4-2(p+q+r)x^2-8p^{1/2}q^{1/2}r^{1/2}x+(p+q+r)^2-4(pq+qr+rp)=0.$

Comparing the coefficients, we have

$$p+q+r=\tfrac{3}{2},$$
$$p^{1/2}q^{1/2}r^{1/2}=\tfrac{3}{2},$$
$$(p+q+r)^2-4(pq+qr+rp)=-2.$$

Therefore $pq+qr+rp=\frac{17}{16}$.

Hence p,q,r are the roots of the cubic $16t^3-24t^2+17t-9=0$.

The roots of the equation are $\frac{1+2\sqrt{2}i}{4}, \frac{1-2\sqrt{2}i}{4}, 1$.

Let $p=\frac{1+2\sqrt{2}i}{4}, q=\frac{1-2\sqrt{2}i}{4}, r=1$.

Then $p^{1/2}=\pm\frac{\sqrt{2}+i}{2}, q^{1/2}=\pm\frac{\sqrt{2}-i}{2}, r^{1/2}=\pm1$.

But $p^{1/2}q^{1/2}r^{1/2}=\frac{3}{4}$.

Therefore the roots are $\frac{\sqrt{2}+i}{2}+\frac{\sqrt{2}-i}{2}+1, \frac{\sqrt{2}+i}{2}-\frac{\sqrt{2}-i}{2}-1, -\frac{\sqrt{2}+i}{2}+\frac{\sqrt{2}-i}{2}-1$ and $-\frac{\sqrt{3}+i}{2}-\frac{\sqrt{2}-i}{2}+1$.

i.e., $\sqrt{2}+1, i-1, -i-1$ and $-\sqrt{2}+1$.

3. (i) If a biquadratic equation has all its roots real, prove that the roots of the Euler's cubic are all non-negative real.

(ii) If a biquadratic equation has two real and two imaginary roots, prove that the Euler's cubic has two imaginary and one non-negative real root.

(iii) If a biquadratic equation has all its roots imaginary, prove that the roots of the Euler's cubic are all real, at least one being negative.

Let $\alpha,\beta,\gamma,\delta$ be the roots of the biquadratic eqaution $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ and p,q,r be the roots of the Euler's cubic. Then

$$p=\tfrac{a_0{}^2}{16}(\beta+\gamma-\alpha-\delta)^2,$$
$$q=\tfrac{a_0{}^2}{16}(\gamma+\alpha-\beta-\delta)^2,$$
$$r=\tfrac{a_0{}^2}{16}(\alpha+\beta-\gamma-\delta)^2.$$

(i) If $\alpha,\beta,\gamma,\delta$ be all real then $p\geq0, q\geq0, r\geq0$.

(ii) Let $\alpha=a+ib, \beta=a-ib$ and γ,δ are real. Then

$p=\frac{a_0{}^2}{16}(-2ib+\gamma-\delta)^2$, $q=\frac{a_0{}^2}{16}(2ib+\gamma-\delta)^2$, $r=\frac{a_0{}^2}{16}(2a-\gamma-\delta)^2$.

Therefore p,q are imaginary, $r\geq0$.

(iii) Let $\alpha = a + ib, \beta = a - ib, \gamma = c + id, \delta = c - id$ where a, b, c, d are real, $b \neq 0, d \neq 0$. Then

$p = -\frac{{a_0}^2}{4}(b-d)^2$, $q = -\frac{{a_0}^2}{4}(b+d)^2$, $r = \frac{{a_0}^2}{4}(a-c)^2$.

Therefore $p \leq 0, q \leq 0, r \geq 0$.

At least one of p and q is negative, because $p = q = 0 \Rightarrow b = d = 0$, a contradiction.

5.12.3. Descartes' method of solution of a biquadratic equation.

Let the biquadratic equation be taken in the standard form

$$x^4 + 6Hx^2 + 4Gx + ({a_0}^2 I - 3H^2) = 0 \quad \cdots \quad \text{(i)}$$

To solve the equation, let us express the left hand expression as the product of two factors in the form

$$(x^2 + lx + m)(x^2 - lx + n).$$

Comparing with the left hand expression of (i) we have

$$m + n - l^2 = 6H \quad \text{or,} \quad m + n = 6H + l^2 \quad \cdots \quad \text{(ii)}$$
$$l(n - m) = 4G \quad \text{or,} \quad m - n = -\frac{4G}{l} \quad \cdots \quad \text{(iii)}$$
$$mn = ({a_0}^2 I - 3H^2) \quad \cdots \quad \text{(iv)}$$

Eliminating m, n we have

$$(6H + l^2)^2 - \frac{16G^2}{l^2} = 4({a_0}^2 - 3H^2) \quad \cdots \quad \text{(v)}$$

This is a cubic equation in l^2 giving l. Then m and n can be determined and finally the solution of the equation (i) is obtained.

Note. Unless the cubic equation (v) has rational roots, the numerical work for finding the roots of the equation (i) becomes laborious.

Worked Example.

1. Solve the equation $x^4 - 3x^2 - 4x - 3 = 0$ by Descartes' method.

Let $x^4 - 3x^2 - 4x - 3 = (x^2 + lx + m)(x^2 - lx + n)$. Then

$$m + n - l^2 = -3 \quad \text{or,} \quad m + n = l^2 - 3 \quad \cdots \text{(i)}$$
$$l(n - m) = -4 \quad \text{or,} \quad m - n = \frac{4}{l} \quad \cdots \text{(ii)}$$
$$mn = -3 \quad \cdots \quad \text{(iii)}$$

Eliminating m, n, we have

$(l^2 - 3)^2 - \frac{16}{l^2} = -12$

or, $\lambda(\lambda - 3)^2 + 12\lambda - 16 = 0$, taking $\lambda = l^2$

or, $\lambda^3 - 6\lambda^2 + 21\lambda - 16 = 0$ ... (iv)

$\lambda = 1$ is a root of the equation (iv).

Taking $l = 1$, we have $m = 1, n = -3$.

The given equation takes the form $(x^2 + x + 1)(x^2 - x - 3) = 0$.

$x^2 + x + 1 = 0$ gives $x = \frac{-1 \pm \sqrt{3}i}{2}$, $x^2 - x - 3 = 0$ gives $x = \frac{1 \pm \sqrt{13}}{2}$.

The roots of the equation are $\frac{-1 \pm \sqrt{3}i}{2}, \frac{1 \pm \sqrt{13}}{2}$.

Note. Taking $l = -1$, we get the same roots of the equation.

Exercises 5G

1. Solve the equations by Ferrari's method

(i) $x^4 + 12x - 5 = 0$,

(ii) $x^4 + 32x - 60 = 0$,

(iii) $x^4 + 3x + 20 = 0$,

(iv) $x^4 - 6x^2 + 16x - 15 = 0$,

(v) $x^4 - 2x^2 + 8x - 3 = 0$,

(vi) $x^4 + 6x^2 + 3x + 10 = 0$,

(vii) $x^4 + 11x^2 + 10x + 50 = 0$,

(viii) $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$,

(ix) $x^4 + 4x^3 - 6x^2 + 20x + 8 = 0$,

(x) $x^4 + 12x^3 + 54x^2 + 96x + 40 = 0$,

(xi) $x^4 + 3x^3 + 5x^2 + 4x + 2 = 0$,

(xii) $2x^4 + 6x^3 - 3x^2 + 2 = 0$.

(xiii) $x^4 - 4x^3 + 5x + 220$

2. Solve the equations by Euler's method

(i) $x^4 + 12x - 5 = 0$,

(ii) $x^4 - 2x^2 + 8x - 3 = 0$,

(iii) $x^4 - 8x^2 - 4x + 3 = 0$,

(iv) $x^4 - 12x^2 + 4 = 0$,

(v) $x^4 - 6x^2 - 16x - 15 = 0$.

3. Solve the equations by Descartes' method

(i) $x^4 - 3x^2 - 6x - 2 = 0$,

(ii) $x^4 - 6x^2 + 16x - 15 = 0$,

(iii) $x^4 - 2x^2 + 8x - 3 = 0$.

4. If the biquadratic equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ has two equal roots, prove that two roots of the reducing cubic are equal. Deduce that in this case $I^3 = 27J^2$.

5. If the biquadratic equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ has three equal roots, prove that all roots of the reducing cubic are zero. Deduce that in this case $I = 0$ and $J = 0$.

6. If the biquadratic equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ has two distinct pairs of equal roots, prove that two roots of the Euler's cubic are zero. Deduce that the equal roots are $\frac{-a_1 \pm \sqrt{3(a_1{}^2 - a_0a_2)}}{a_0}$.

6. SUMMATION OF SERIES

6.1. Introduction.

Let $u_1 + u_2 + \cdots + u_n + \cdots$ be a series where u_n is some function of the positive integral variable n. The sum of the first n terms $u_1 + u_2 + \cdots + u_n$ is denoted as $\sum_{r=1}^{n} u_r$, or as $\sum_{1}^{n} u_r$. The sequence $\{s_n\}$ where $s_n = \sum_{r=1}^{n} u_r$ is said to be the *sequence of partial sums* of the series. If the sequence $\{s_n\}$ converges to a finite limit s then the series $u_1 + u_2 + \cdots + u_n + \cdots$ is said to be *convergent* and in that case s is said to be the *sum* of the series. If the sequence $\{s_n\}$ is *divergent*, the series is said to be *divergent*.

We shall discuss in this chapter some methods of finding the sum s_n for the series $u_1 + u_2 + \cdots + u_n + \cdots$

6.2. Method of difference.

Let $u_1 + u_2 + u_3 + \cdots$ be a given series. If we can express u_r as $u_r = v_r - v_{r-1}$ where v_r is some function of r, then $\sum_{1}^{n} u_r$ can be readily obtained.

Here $u_1 = v_1 - v_0$

$u_2 = v_2 - v_1$

$\cdots \quad \cdots$

$u_n = v_n - v_{n-1}$.

By addition, $\sum_{1}^{n} u_r = v_n - v_0$.

Worked Examples.

1. Find the sum upto n terms $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots\cdots$

Let the series be denoted by $u_1 + u_2 + u_3 + \cdots\cdots$

Then $u_r = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$.

Therefore $u_1 = 1 - \frac{1}{2}$

$u_2 = \frac{1}{2} - \frac{1}{3}$

$\cdots \quad \cdots$

$u_n = \frac{1}{n} - \frac{1}{n+1}$.

By addition, $\sum_1^n u_r = 1 - \frac{1}{n+1} = \frac{n}{n+1}$.

2. Find the sum upto n terms $\quad 1.1! + 2.2! + 3.3! + \cdots \cdots$

Denoting the series by $u_1 + u_2 + u_3 + \cdots \cdots$, we have

$u_r = r.r! = (r+1).r! - r!$

$= (r+1)! - r! = v_{r+1} - v_r$, where $v_r = r!$.

Therefore $\sum_1^n u_r = v_{n+1} - v_1 = (n+1)! - 1$.

3. Find the sum upto n terms $\quad a + ax + ax^2 + \cdots \cdots$ where $x \neq 1$.

Denoting the series by $u_1 + u_2 + u_3 + \cdots \cdots$, we have

$u_r = ax^{r-1} = a\frac{x^r - x^{r-1}}{x-1}$

$= v_r - v_{r-1}$, where $v_r = \frac{a}{x-1}x^r$.

Therefore $\sum_1^n u_r = v_n - v_0 = \frac{a(x^n - 1)}{x - 1}$.

The method of difference can be effectively applied to the following types of series.

A. To find the sum of n terms of a series each term of which is composed of m factors in some arithmetic progression, the first factors of successive terms being in the same arithmetic progression.

Let the series be $u_1 + u_2 + \cdots + u_n + \cdots$, where

$u_r = (a+rb)(a+\overline{r+1}b)\cdots(a+\overline{r+m-1}b)$.

u_r can be expressed as

$u_r = (a+rb)(a+\overline{r+1}b)\cdots(a+\overline{r+m-1}b)\frac{\{(a+\overline{r+m}b)-(a+\overline{r-1}b)\}}{(m+1)b}$

$= v_r - v_{r-1}$,

where $v_r = \frac{1}{(m+1)b}(a+rb)(a+\overline{r+1}b)\cdots(a+\overline{r+m}b)$.

Therefore $u_1 = v_1 - v_0$

$u_2 = v_2 - v_1$

$\cdots \quad \cdots$

$u_n = v_n - v_{n-1}$.

By addition, $\sum_1^n u_r = v_n - v_0$

$= \frac{1}{(m+1)b}(a+nb)(a+\overline{n+1}b)\cdots(a+\overline{n+m}b) - v_0$

$= \frac{u_n(a+\overline{n+m}b)}{(m+1)b} + c$, where c is independent of n.

Working rule to find $\sum_1^n u_r$.

Write down u_n. Introduce the next factor of the arithmetic progression at the end. Divide by the number of factors thus increased and the common difference. Then add a constant. The constant c can be determined by giving a particular value to n.

Worked Examples (continued).

4. Find the sum upto n terms $\quad 1.3 + 3.5 + 5.7 + \cdots \cdots$

Let $u_1 + u_2 + u_3 + \cdots \cdots$ be the given series.

Then $u_r = (2r-1)(2r+1)$.

Hence $\sum_1^n u_r = \frac{(2n-1)(2n+1)(2n+3)}{3.2} + c$, where c is a constant.

But $u_1 = 1.3$. Therefore $3 = \frac{1.3.5}{3.2} + c$. This gives $c = \frac{1}{2}$.

Therefore $\sum_1^n u_r = \frac{(2n-1)(2n+1)(2n+3)}{6} + \frac{1}{2}$.

5. Find the sum upto n terms $\quad 1.3 + 2.4 + 3.5 + \cdots \cdots$

Denoting the series by $u_1 + u_2 + u_3 + \cdots \cdots$, we have

$u_r = r(r+2)$.

Here the rule is not applicable. Each term of the series is the product of two factors in an arithmetic progression but the first factors of successive terms are not in the same arithmetic progression.

u_r can be expressed as $\quad u_r = r(r+1+1) \quad = r(r+1) + r$.

Hence $\sum_1^n u_r = \frac{n(n+1)(n+2)}{3.1} + \frac{n(n+1)}{2.1} + c$, where c is a constant.

But $u_1 = 1.3$. Therefore $3 = \frac{1.2.3}{3} + \frac{1.2}{2} + c$. This gives $c = 0$.

Therefore $\sum_1^n u_r = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2}$.

6. Find the sum upto n terms $\quad 1^2 + 2^2 + 3^2 + \cdots \cdots$

Denoting the series by $u_1 + u_2 + u_3 + \cdots \cdots$, we have

$u_r = r^2 = r(r+1-1) = r(r+1) - r$.

Therefore $\sum_1^n u_r = \frac{n(n+1)(n+2)}{3.1} - \frac{n(n+1)}{2.1} + c$, where c is a constant.

But $u_1 = 1$. Therefore $1 = \frac{1.2.3}{3} + \frac{1.2}{2} + c$. This gives $c = 0$.

Therefore $\sum_1^n u_r = \frac{n(n+1)(n+2)}{3.1} - \frac{n(n+1)}{2.1} = \frac{n(n+1)(2n+1)}{6}$.

B. To find the sum of n terms of a series each term of which is the reciprocal of m factors in arithmetic progression, the first factors of successive terms being in the same arithmetic progression.

Let the series be $u_1 + u_2 + \cdots + u_n + \cdots$, where

$u_r = \frac{1}{(a+rb)(a+r+1b)\cdots(a+r+m-1b)}$.

u_r can be expressed as

$u_r = \frac{1}{(a+rb)\cdots\cdots(a+r+m-1b)}\left\{\frac{(a+r+m-1b)-(a+rb)}{(m-1)b}\right\} = v_r - v_{r+1}$,

where $v_r = \frac{1}{(m-1)b}\cdot\frac{1}{(a+rb)(a+r+1b)\cdots(a+r+m-2b)}$.

Therefore $u_1 = v_1 - v_2$

$u_2 = v_2 - v_3$

$\cdots \quad \cdots$

$u_n = v_n - v_{n+1}$.

By addition, $\sum_1^n u_r = v_1 - v_{n+1}$

$= c - \frac{1}{(m-1)b}\cdot\frac{1}{(a+n+1b)(a+n+2b)\cdots(a+n+m-1b)}$

$= c - \frac{1}{(m-1)b}\cdot u_n(a+nb)$, where c is independent of n.

Working rule to find $\sum_1^n u_r$.

Write down u_n. Delete a factor from the beginning. Divide by the number of factors thus diminished and the common difference. Then change the sign and add a constant. The constant c can be determined by assigning to n some particular value.

Worked Examples (continued).

7. Find the sum to n terms of the series

$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots\cdots$

Denoting the series by $u_1 + u_2 + u_3 + \cdots\cdots$, we have

$u_r = \frac{1}{r(r+1)(r+2)}$.

Hence $\sum_1^n u_r = c - \frac{1}{2.1.(n+1)(n+2)}$, where c is a constant.

But $u_1 = \frac{1}{1.2.3} = \frac{1}{6}$. Therefore $\frac{1}{6} = c - \frac{1}{2.2.3}$. This gives $c = \frac{1}{4}$.

Therefore $\sum_1^n u_r = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$.

8. Find the sum to n terms of the series $\frac{1}{1.5} + \frac{1}{3.7} + \frac{1}{5.9} + \cdots\cdots$

Denoting the series by $u_1 + u_2 + u_3 + \cdots\cdots$, we have

$u_r = \frac{1}{(2r-1)(2r+3)}$.

The rule is not directly applicable here. Each term of the series is the reciprocal of the product of two factors in an arithmetic progression but the first factors of several terms are in a different arithmetic progression.

u_r can be expressed as

$u_r = \frac{2r+1}{(2r-1)(2r+1)(2r+3)}$

$= \frac{(2r-1)+2}{(2r-1)(2r+1)(2r+3)}$

$= \frac{1}{(2r+1)(2r+3)} + \frac{2}{(2r-1)(2r+1)(2r+3)}$.

Therefore $\sum_1^n u_r = c - \frac{1}{1.2(2n+3)} - \frac{2}{2.2(2n+1)(2n+3)}$, where c is a constant.

But $u_1 = \frac{1}{5}$. Therefore $\frac{1}{5} = c - \frac{1}{2.5} - \frac{2}{4.3.5}$. This gives $c = \frac{1}{3}$.

Therefore $\sum_1^n u_r = \frac{1}{3} - \frac{1}{2(2n+3)} - \frac{1}{2(2n+1)(2n+3)}$.

9. Find the sum to n terms

$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots\cdots$

Denoting the series by $u_1 + u_2 + u_3 + \cdots\cdots$, we have

$u_r = \frac{2r-1}{r(r+1)(r+2)} = \frac{2}{(r+1)(r+2)} - \frac{1}{r(r+1)(r+2)}$.

Therefore $\sum_1^n u_r = c - \frac{2}{n+2} + \frac{1}{2(n+1)(n+2)}$, where c is a constant.

But $u_1 = \frac{1}{1.2.3} = \frac{1}{6}$. Therefore $\frac{1}{6} = c - \frac{2}{3} + \frac{1}{2.2.3}$. This gives $c = \frac{3}{4}$.

Therefore $\sum_1^n u_r = \frac{3}{4} - \frac{2}{n+2} + \frac{1}{2(n+1)(n+2)}$.

C. To find the sum of n terms of the series $u_1 + u_2 + \cdots + u_n + \cdots$

where $u_r = \frac{a(a+d)(a+2d)\cdots(a+r-1d)}{b(b+d)(b+2d)\cdots(b+r-1d)}$.

The series is $\frac{a}{b} + \frac{a(a+d)}{b(b+d)} + \frac{a(a+d)(a+2d)}{b(b+d)(b+2d)} + \cdots\cdots$

u_r can be expressed as

$u_r = \frac{a(a+d)\cdots(a+r-1d)}{b(b+d)\cdots(b+r-1d)}\left\{\frac{(a+rd)-(b+r-1d)}{a-b+d}\right\}$

$$= \frac{1}{a-b+d}\cdot\frac{a(a+d)\cdots(a+rd)}{b(b+d)\cdots(b+\overline{r-1}d)} - \frac{1}{a-b+d}\cdot\frac{a(a+d)\cdots(a+\overline{r-1}d)}{b(b+d)\cdots(b+\overline{r-2}d)}$$

$= v_r - v_{r-1}$, where $v_r = \frac{1}{a-b+d}\cdot\frac{a(a+d)\cdots(a+rd)}{b(b+d)\cdots(b+\overline{r-1}d)}$.

Therefore $u_1 = v_1 - v_0$
$u_2 = v_2 - v_1$
$\cdots \quad \cdots$
$u_n = v_n - v_{n-1}$

By addition, $\sum_1^n u_r = v_n - v_0$.

Worked Example (continued).

10. Find the sum to n terms
$\frac{4}{5} + \frac{4.7}{5.8} + \frac{4.7.10}{5.8.11} + \cdots\cdots$

Denoting the series by $u_1 + u_2 + u_3 + \cdots\cdots$, we have

$u_r = \frac{4.7\cdots(3r+1)}{5.8\cdots(3r+2)} = \frac{4.7\cdots(3r+1)}{5.8\cdots(3r+2)}\left\{\frac{(3r+4)-(3r+2)}{2}\right\}$

$= \frac{1}{2}\cdot\frac{4.7\cdots(3r+4)}{5.8\cdots(3r+2)} - \frac{1}{2}\cdot\frac{4.7\cdots(3r+1)}{5.8\cdots(3r-1)}$

$= v_r - v_{r-1}$, where $v_r = \frac{4.7\cdots(3r+4)}{2.5\cdots(3r+2)}$.

Therefore $\sum_1^n u_r = v_n - v_0 = \frac{4.7\cdots(3n+4)}{2.5\cdots(3n+2)} - 2$.

D. Application of the method of difference to find the sum to n terms of some Trigonometrical series.

(i) Find the sum of first n terms of the series
$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots\cdots$

Here the angles are in arithmetic progression, the first angle being α and the common difference being β.

Here $u_r = \sin(\alpha + \overline{r-1}\beta)$

$= \frac{2\sin(\alpha+\overline{r-1}\beta)\sin\frac{\beta}{2}}{2\sin\frac{\beta}{2}}$

$= \frac{\cos(\alpha+\frac{2r-3}{2}\beta)-\cos(\alpha+\frac{2r-1}{2}\beta)}{2\sin\frac{\beta}{2}}$

$= v_r - v_{r+1}$, where $v_r = \frac{\cos(\alpha+\frac{2r-3}{2}\beta)}{2\sin\frac{\beta}{2}}$.

Therefore $\sum_1^n u_r = v_1 - v_{n+1}$

$= \frac{\cos(\alpha-\frac{\beta}{2})-\cos(\alpha+\frac{2n-1}{2}\beta)}{2\sin\frac{\beta}{2}}$

$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin(\alpha+\frac{n-1}{2}\beta)$

$= \frac{\sin\frac{n\ \text{diff}}{2}}{\sin\frac{\text{diff}}{2}}\sin(\frac{\text{first angle + last angle}}{2})$.

(ii) Find the sum of first n terms of the series
$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots\cdots$

Here the angles are in arithmetic progression, the first angle being α and the common difference being β.

Here $u_r = \cos(\alpha + \overline{r-1}\beta)$

$= \frac{2\sin\frac{\beta}{2}\cos(\alpha+\overline{r-1}\beta)}{2\sin\frac{\beta}{2}}$

$= \frac{\sin(\alpha+\frac{2r-1}{2}\beta)-\sin(\alpha+\frac{2r-3}{2}\beta)}{2\sin\frac{\beta}{2}}$

$= v_{r+1} - v_r$ where $v_r = \frac{\sin(\alpha+\frac{2r-3}{2}\beta)}{2\sin\frac{\beta}{2}}$.

Therefore $\sum_1^n u_r = v_{n+1} - v_1$

$= \frac{\sin(\alpha+\frac{2n-1}{2}\beta)-\sin(\alpha-\frac{\beta}{2})}{2\sin\frac{\beta}{2}}$

$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos(\alpha+\frac{n-1}{2}\beta)$

$= \frac{\sin\frac{n\ \text{diff.}}{2}}{\sin\frac{\text{diff.}}{2}}\cos(\frac{\text{first angle + last angle}}{2})$.

(iii) Find the sum to n terms $\quad \text{cosec}\,\theta + \text{cosec}\,2\theta + \text{cosec}\,2^2\theta + \cdots\cdots$

We have $\cot\theta - \cot 2\theta$

$= \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{\sin(2\theta-\theta)}{\sin\theta\sin 2\theta} = \text{cosec}\,2\theta$.

Therefore $\text{cosec}\,\theta = \cot\frac{\theta}{2} - \cot\theta$
$\text{cosec}\,2\theta = \cot\theta - \cot 2\theta$
$\text{cosec}\,2^2\theta = \cot 2\theta - \cot 2^2\theta$
$\cdots \quad \cdots \quad \cdots$
$\text{cosec}\,2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta$.

By addition,
$\text{cosec}\,\theta + \text{cosec}\,2\theta + \cdots + \text{cosec}\,2^{n-1}\theta = \cot\frac{\theta}{2} - \cot 2^{n-1}\theta$.

(iv) Find the sum to n terms $\tan\theta + 2\tan 2\theta + 2^2\tan 2^2\theta + \cdots$

we have $\cot\theta - \tan\theta = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \frac{2\cos 2\theta}{\sin 2\theta} = 2\cot 2\theta.$

Therefore $\tan\theta = \cot\theta - 2\cot 2\theta.$

$2\tan 2\theta = 2\cot 2\theta - 2^2\cot 2^2\theta$

$2^2\tan 2^2\theta = 2^2\cot 2^2\theta - 2^3\cot 2^3\theta$

... ...

$2^{n-1}\tan 2^{n-1}\theta = 2^{n-1}\cot 2^{n-1}\theta - 2^n\cot 2^n\theta.$

By addition,

$\tan\theta + 2\tan 2\theta + \cdots + 2^{n-1}\tan 2^{n-1}\theta = \cot\theta - 2^n\cot 2^n\theta.$

(v) Find the sum to n terms

$\tan^{-1}\frac{1}{1+1.2} + \tan^{-1}\frac{1}{1+2.3} + \tan^{-1}\frac{1}{1+3.4} + \cdots \cdots$

Here $u_r = \tan^{-1}\frac{1}{1+r(r+1)}$

$= \tan^{-1}\frac{(r+1)-r}{1+(r+1)r}$

$= \tan^{-1}(r+1) - \tan^{-1} r$

$= v_{r+1} - v_r$ where $v_r = \tan^{-1} r.$

Therefore $\sum_1^n u_r = \tan^{-1}(n+1) - \tan^{-1} 1.$

(vi) Find the sum to n terms

$\frac{1}{\sin\theta\sin 2\theta} + \frac{1}{\sin 2\theta\sin 3\theta} + \frac{1}{\sin 3\theta\sin 4\theta} + \cdots \cdots$

Here $u_r = \frac{1}{\sin r\theta\sin(r+1)\theta}$

$= \frac{\sin\{(r+1)\theta - r\theta\}}{\sin\theta\sin r\theta\sin(r+1)\theta}$

$= \frac{1}{\sin\theta}\{\cot r\theta - \cot(r+1)\theta\}$

$= v_r - v_{r+1}$ where $v_r = \frac{\cot r\theta}{\sin\theta}.$

Therefore $\sum_1^n u_r = v_1 - v_{n+1} = \{\cot\theta - \cot(n+1)\theta\}\operatorname{cosec}\theta.$

(vii) Find the sum to n terms

$\tan\theta\sec 2\theta + \tan 2\theta\sec 2^2\theta + \tan 2^2\theta\sec 2^3\theta + \cdots \cdots$

Here $u_r = \tan 2^{r-1}\theta\sec 2^r\theta$

$= \frac{\sin 2^{r-1}\theta}{\cos 2^{r-1}\theta\cos 2^r\theta}$

$= \frac{\sin(2^r\theta - 2^{r-1}\theta)}{\cos 2^{r-1}\theta\cos 2^r\theta}$

$= \tan 2^r\theta - \tan 2^{r-1}\theta.$

Therefore $u_1 = \tan 2\theta - \tan\theta$

$u_2 = \tan 2^2\theta - \tan 2\theta$

.....

$u_n = \tan 2^n\theta - \tan 2^{n-1}\theta.$

By addition, $\sum_1^n u_r = \tan 2^n\theta - \tan\theta.$

Exercises 6A

1. Sum the series to n terms

(i) $1.2.3 + 2.3.4 + 3.4.5 + \cdots \cdots$

(ii) $2.3.4.5. + 3.4.5.6 + 4.5.6.7 + \cdots \cdots$

(iii) $2.5.8. + 5.8.11 + 8.11.14 + \cdots \cdots$

(iv) $1.2^2 + 2.3^2 + 3.4^2 + \cdots \cdots$

(v) $1.4 + 2.5 + 3.6 + \cdots \cdots$

(vi) $1.2.4^2 + 2.3.5^2 + 3.4.6^2 + \cdots \cdots$

(vii) $1.3.5 + 2.4.6 + 3.5.7 + \cdots \cdots$

(viii) $1.2 + 3.4 + 5.6 + \cdots \cdots$

2. Sum the series to n terms

(i) $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots \cdots$

(ii) $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \cdots \cdots$

(iii) $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + \cdots \cdots$

(iv) $\frac{2}{1.3.4} + \frac{3}{2.4.5} + \frac{4}{3.5.6} \cdots \cdots$

(v) $\frac{1}{2.3.4} + \frac{3}{3.4.5} + \frac{5}{4.5.6} + \cdots \cdots$

(vi) $\frac{2}{3.5} + \frac{2.4}{3.5.7} + \frac{2.4.6}{3.5.7.9} + \cdots \cdots$

(vii) $\frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \frac{1.3.5.7}{2.4.6.8} + \cdots \cdots$

(viii) $\frac{2}{4} + \frac{2.5}{4.7} + \frac{2.5.8}{4.7.10} + \cdots \cdots$

3. Sum the series to n terms

(i) $\sin\theta - \sin 2\theta + \sin 3\theta - \cdots \cdots$

(ii) $\sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + \cdots \cdots$

(iii) $\cos^2\theta + \cos^2 3\theta + \cos^2 5\theta + \cdots$ ...

(iv) $\sin\theta \sin 2\theta + \sin 2\theta \sin 3\theta + \sin 3\theta \sin 4\theta + \cdots$ ...

(v) $\cos\theta \cos 2\theta + \cos 3\theta \cos 4\theta + \cos 5\theta \cos 6\theta + \cdots$ ...

(vi) $\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \cdots$ ...

(vii) $\tan x \tan 2x + \tan 2x \tan 3x + \tan 3x \tan 4x + \cdots$ ...

(viii) $\sec\theta \sec 2\theta + \sec 2\theta \sec 3\theta + \sec 3\theta \sec 4\theta + \cdots$ ...

(ix) $2\text{cosec } 2\theta \cot 2\theta + 4\text{cosec } 4\theta \cot 4\theta + 8\text{cosec } 8\theta \cot 8\theta + \cdots \cdots$

(x) $\frac{\sin\theta}{\cos\theta+\cos 2\theta} + \frac{\sin 2\theta}{\cos\theta+\cos 4\theta} + \frac{\sin 3\theta}{\cos\theta+\cos 6\theta} + \cdots \cdots$

(xi) $\log(2\cos\theta) + \log(2\cos 2\theta) + \log(2\cos 2^2\theta) + \cdots \cdots$

(xii) $\sin x \sec 3x + \sin 3x \sec 3^2 x + \sin 3^2 x \sec 3^3 x + \cdots \cdots$

(xiii) $\tan^{-1}\frac{1}{2.1^2} + \tan^{-1}\frac{1}{2.2^2} + \tan^{-1}\frac{1}{2.3^2} + \cdots \cdots$

(xiv) $\cot^{-1}(1+1+1^2) + \cot^{-1}(1+2+2^2) + \cdots \cdots$

(xv) $\cot^{-1}(2+\frac{1.2}{2}) + \cot^{-1}(2+\frac{2.3}{2}) + \cot^{-1}(2+\frac{3.4}{2}) + \cdots$ ...

(xvi) $\log(1+2\cos\theta) + \log(1+2\cos 3\theta) + \log(1+2\cos 3^2\theta) + \cdots \cdots$

4. Show that

(i) $1.50 + 2.49 + 3.48 + \cdots + 50.1 = 22100,$

(ii) $1.50^2 + 2.49^2 + 3.48^2 + \cdots + 50.1^2 = 563500,$

(iii) $1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \cdots + \frac{50}{3^{49}} = \frac{9(3^{50}-1)-300}{4.3^{50}},$

[**Hint.** Let $s = 1 + 2x + \cdots + 50x^{49}$, where $x = \frac{1}{3}$. Find $s - sx$.]

(iv) $1.n + 2(n-1) + 3(n-2) + \cdots + n.1 = \frac{1}{12}n(n+1)^2(n+2),$

(v) $\frac{n}{1.2.3} + \frac{n-1}{2.3.4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+1)}{4(n+2)},$

(vi) $1 + 2(1+\frac{1}{p}) + 3(1+\frac{1}{p^2}) + 4(1+\frac{1}{p^3}) + \cdots + n(1+\frac{1}{p^{n-1}})$

$= \frac{p(p^n-1)-n(p-1)}{(p-1)^2 p^{n-1}} + \frac{n(n+1)}{2}.$

6.3. Difference operator Δ.

Let us consider a sequence $\{u_1, u_2, \ldots u_n, \ldots \ldots\}$ where u_n is a function of positive integral variable n.

We define $\Delta u_n = u_{n+1} - u_n$.

Then $\Delta u_1 = u_2 - u_1, \Delta u_2 = u_3 - u_2, \Delta u_3 = u_4 - u_3, \ldots \ldots$

The sequence $\{\Delta u_1, \Delta u_2, \ldots \Delta u_n, \ldots \ldots\}$ is said to be the *first order of differences* of the sequence $\{u_n\}$.

We define $\Delta^2 u_n = \Delta(\Delta u_n) = \Delta(u_{n+1} - u_n)$.

The sequence $\{\Delta^2 u_1, \Delta^2 u_2, \Delta^2 u_3, \ldots \ldots\}$ is said to be the *second order of differences* of the sequence $\{u_n\}$.

In a similar manner the pth order of differences $\Delta^p u_n$ where p is a positive integer, can be defined for the sequence $\{u_n\}$.

It follows immediately that

(i) if $u_n = k$, a constant for all n then $\Delta u_n = 0$ for all n, and conversely;

(ii) If $\{u_n\}$ be a sequence in an arithmetic progression having the common difference d then $\Delta u_n =$ constant $= d$ for all n and conversely.

(iii) If $\{u_n\}$ be a sequence in a geometric progression having the common ratio r then the sequence $\{\Delta u_n\}$ is also in geometric progression of common ratio r.

Examples.

1. Let the sequence be $8, 18, 34, 56, 84, 118, \ldots \ldots$

The successive orders of differences are

10	16	22	28	34	...
6	6	6	6	...	...
0	0	0	...	...	...

Here the second order of differences is a sequence of equal elements. The third and the successive orders of differences are sequences of zeros.

2. Let the sequence be $4, 12, 36, 108, 324, 972, \ldots \ldots$:

The successive orders of differences are

8	24	72	216	648	...
16	48	144	432	...	...

Here the given sequence is in a geometric progression of common ratio 3. The successive orders of differences are also in geometric progression of common ratio 3.

Let us consider the sequence $\{u_1, u_2, u_3, \ldots \ldots\}$ and the successive orders of differences.

u_1	u_2	u_3	u_4	u_5	...	
Δu_1	Δu_2	Δu_3	Δu_4	...	...	
$\Delta^2 u_1$	$\Delta^2 u_2$	$\Delta^2 u_3$	...	...	...	
$\Delta^3 u_1$	$\Delta^3 u_2$	...	...	...	...	
...	...		...			

It appears from the table that

$$\begin{aligned} u_2 &= u_1 + \Delta u_1, \\ u_3 &= u_2 + \Delta u_2 \\ &= (u_1 + \Delta u_1) + \Delta(u_1 + \Delta u_1) \\ &= u_1 + 2\Delta u_1 + \Delta^2 u_1, \end{aligned}$$

$$\begin{aligned} u_4 &= u_3 + \Delta u_3 \\ &= (u_2 + \Delta u_2) + \Delta(u_2 + \Delta u_2) \\ &= u_2 + 2\Delta u_2 + \Delta^2 u_2 \\ &= (u_1 + \Delta u_1) + 2\Delta(u_1 + \Delta u_1) + \Delta^2(u_1 + \Delta u_1) \\ &= u_1 + 3\Delta u_1 + 3\Delta^2 u_1 + \Delta^3 u_1, \\ &\quad \cdots \quad \cdots \quad \cdots \end{aligned}$$

Theorem 6.3.1. Let $\{u_n\}$ be a sequence and Δ^p denote the pth order difference operator. Then

$$u_n = u_1 + \binom{n-1}{1}\Delta u_1 + \binom{n-1}{2}\Delta^2 u_1 + \cdots + \binom{n-1}{r}\Delta^r u_1 + \cdots + \Delta^{n-1}u_1.$$

Proof. We prove the theorem by the method of induction.

The theorem is obviously true for $n = 1$.

Let us assume the theorem to be true for $n = m$, where m is a positive integer.

Then $u_m = u_1 + \binom{m-1}{1}\Delta u_1 + \binom{m-1}{2}\Delta^2 u_1 + \cdots + \binom{m-1}{r}\Delta^r u_1 + \cdots + \Delta^{m-1}u_1$.

Therefore $\Delta u_m = \Delta u_1 + \binom{m-1}{1}\Delta^2 u_1 + \binom{m-1}{2}\Delta^3 u_1 + \cdots + \binom{m-1}{r}\Delta^{r+1}u_1 + \cdots + \Delta^m u_1$.

$$\begin{aligned} \text{Hence } u_{m+1} &= u_m + \Delta u_m \\ &= u_1 + [1 + \binom{m-1}{1}]\Delta u_1 + [\binom{m-1}{1} + \binom{m-1}{2}]\Delta^2 u_1 + \\ &\quad + [\binom{m-1}{r-1} + \binom{m-1}{r}]\Delta^r u_1 + \cdots + \Delta^m u_1 \\ &= u_1 + \binom{m}{1}\Delta u_1 + \cdots + \binom{m}{r}\Delta^r u_1 + \cdots + \Delta^m u_1. \end{aligned}$$

This shows that the theorem is true for $n = m + 1$ if it be true for $n = m$. And the theorem is true for $n = 1$.

By the principle of induction, the theorem holds for all positive integers n.

Corollary. If p be the least positive integer such that $\{\Delta^p u_n\}$ is a sequence of equal elements, then u_n is a polynomial in n of degree p. Because in this case all the sequences of $(p+1)$th and the higher orders of differences are sequences of zeros.

$$\begin{aligned} \text{Therefore } u_n &= u_1 + (n-1)\Delta u_1 + \cdots + (n-1)(n-2)\cdots(n-p)\frac{\Delta^p u_1}{p!} \\ &= \text{a polynomial in } n \text{ of degree } p. \end{aligned}$$

Worked Examples.

1. *Find u_n where the sequence* $\{u_n\} = \{3, 10, 21, 36, 55, \ldots\ldots\}$

The successive orders of differences are

7	11	15	19	...
4	4	4	...	...

$$\begin{aligned} \text{Therefore } u_n &= 3 + (n-1)7 + \frac{(n-1)(n-2)}{2!}.4 \\ &= 2n^2 + n. \end{aligned}$$

Alternatively, u_n is a polynomial in n of degree 2, since the second order of differences is a sequence of equal elements.

Let $u_1 = an^2 + bn + c$.

$$\begin{aligned} \text{Since } u_1 &= 3, & a + b + c &= 3; \\ \text{since } u_2 &= 10, & 4a + 2b + c &= 10; \\ \text{since } u_3 &= 21, & 9a + 3b + c &= 21. \end{aligned}$$

These determine $a = 2, b = 1, c = 0$.

Therefore $u_n = 2n^2 + n$.

2. Find the sum to n terms of the series

$3 + 11 + 31 + 69 + 131 + \cdots$

Let the series be $u_1 + u_2 + u_3 + \cdots$

The sequence $\{u_n\}$ and the successive orders of differences are

3	11	31	69	131	...
8	20	38	62	...	...
12	18	24	...	...	...
6	6	...	...	...	...

$$\begin{aligned} \text{Therefore } u_n &= 3 + 8(n-1) + 12.\frac{(n-1)(n-2)}{2} + 6.\frac{(n-1)(n-2)(n-3)}{6} \\ &= n^3 + n + 1. \end{aligned}$$

$$\begin{aligned} \text{Hence } \sum_1^n u_n &= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)}{2} + n \\ &= n[\frac{(n^3+2n^2+n)+(2n+2)+4}{4}] \\ &= n(\frac{n^3+2n^2+3n+6}{4}). \end{aligned}$$

3. Find the nth term and the sum to n terms of the series

$6.3 + 9.8 + 12.15 + 15.24 + 18.35 + \cdots\cdots$

Let $u_1 + u_2 + u_3 + \cdots\cdots$ be the given series. Let $\{v_n\}$ and $\{w_n\}$ be the sequences $\{6, 9, 12, 15, 18, \ldots\ldots\}, \{3, 8, 15, 24, 35, \ldots\ldots\}$ respectively.

The successive orders of differences of the sequence $\{w_n\}$ are

3	8	15	24	35	...
	5	7	9	11	... ...
		2	2	2	... ...

Therefore $w_n = 3 + 5.(n-1) + 2.\frac{(n-1)(n-2)}{2}$
$= n^2 + 2n;$
and $u_n = 3n(n+1)(n+2).$

Hence $\sum_1^n u_r = \frac{3n(n+1)(n+2)(n+3)}{4} + c$ where c is a constant.

But $u_1 = 18$. $18 = \frac{3.2.3.4}{4} + c$ and this gives $c = 0$.

Therefore $\sum_1^n u_r = \frac{3}{4}n(n+1)(n+2)(n+3)$.

4. Sum to n terms the series
$2.1 + 7.2 + 14.2^2 + 23.2^3 + 34.2^4 + \cdots \cdots$

Let $\{v_n\}$ be the sequence $\{2, 7, 14, 23, 34, \cdots \cdots\}$

The successive orders of differences are

5	7	9	11	...
	2	2	2	... ...

Therefore $v_n = 2 + (n-1)5 + \frac{(n-1)(n-2)}{2}.2$
$= n^2 + 2n - 1.$

Let the given series be $u_1 + u_2 + u_3 + \cdots \cdots$
Then $u_n = (n^2 + 2n - 1).2^{n-1}$.

Let $(n^2 + 2n - 1).2^{n-1} = f(n).2^n - f(n-1).2^{n-1}$ where $f(n)$ is a polynomial of degree 2. Let us assume $f(n) = an^2 + bn + c$.

Then $(n^2 + 2n - 1).2^{n-1}$
$= (an^2 + bn + c).2^n - \{a(n-1)^2 + b(n-1) + c\}.2^{n-1}$
$= \{2an^2 + 2bn + 2c - a(n-1)^2 - b(n-1) - c\}.2^{n-1}$
$= \{an^2 + (b+2a)n - a + b + c\}.2^{n-1}$.

Equating, we have $a = 1, b = 0, c = 0$.
So $u_n = n^2.2^n - (n-1)^2.2^{n-1}$.

Therefore $u_1 = 1.2$
$u_2 = 2^2.2^2 - 1.2$
$u_3 = 3^2.2^3 - 2^2.2^2$
$\cdots \quad \cdots \quad \cdots$
$u_n = n^2.2^n - (n-1)^2.2^{n-1}$.

By addition, $\sum_1^n u_r = n^2.2^n$.

Let us consider the sequence $\{u_n\}$ where $u_n = ar^{n-1}$. Then the sequence is $\{a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots \ldots\}$, a geometric progression of common ratio r.

The successive orders of differences are

$a(r-1)$		$ar(r-1)$		$ar^2(r-1)$		...
	$a(r-1)^2$		$ar(r-1)^2$		$ar^2(r-1)^2$	...
		$a(r-1)^3$		$ar(r-1)^3$		...

Here each order of differences is a sequence in geometric progression of common ratio r.

Let us consider the sequence $\{u_n\}$, where $u_n = k + ar^{n-1}$, k being a constant.

Then $\Delta u_n = u_{n+1} - u_n$
$= (k + ar^n) - (k + ar^{n-1})$
$= ar^{n-1}(r-1.)$

This shows that the first order of differences is a sequence in geometric progression of common ratio r. Therefore the successive orders of differences are also sequences in geometric progression of common ratio r.

Let $u_n = \phi(n) + ar^{n-1}$ where $\phi(n)$ is a polynomial of degree 1.

Then $\Delta u_n = \phi(n+1) - \phi(n) + ar^{n-1}(r-1)$
$= k + ar^{n-1}(r-1)$, k being a constant.
$\Delta^2 u_n = ar^{n-1}(r-1)^2$.

Here the second and the successive orders of differences are sequences in geometric progression of common ratio r.

Let $u_n = \phi(n) + ar^{n-1}$, where $\phi(n)$ is a polynomial of degree p.
Then the $(p+1)$th and the successive orders of differences are sequences in geometric progression of common ratio r.

Worked Examples (continued).

5. Find u_n where the sequence $\{u_n\} = \{7, 14, 33, 88, 251, \ldots \ldots\}$

The succesive orders of differences are

7	19	55	163	...
	12	36	108	... ...
		24	72	... ...

Here the second and the successive orders of differences are sequences in geometric progression of common ratio 3. Therefore $u_n = a.3^{n-1} + \phi(n)$ where $\phi(n)$ is a polynomial of degree 1.

Let $u_n = a.3^{n-1} + bn + c$.
Then $a+b+c=7, \quad 3a+2b+c=14, \quad 9a+3b+c=33$.
These determine $a=3, b=1, c=3$.
Therefore $u_n = 3^n + n + 3$.

6. Find the sum of the first n terms of the series

$$7+14+33+88+251+\cdots\cdots$$

By the previous example, the nth term u_n is given by

$$u_n = 3^n + n + 3.$$

$$\text{Therefore } \sum_1^n u_r = 3(1+3+3^3+\cdots+3^{n-1}) + \frac{n(n+1)}{2} + 3n$$
$$= 3.\frac{3^n-1}{2} + \frac{n(n+1)}{2} + 3n.$$

6.4. Recurring series.

The series $u_0 + u_1 + u_2 + \cdots\cdots$ is said to be a *recurring series* of order r if any $r+1$ successive terms of the series are connected by the relation

$$u_n + p_1u_{n-1} + p_2u_{n-2} + \cdots + p_ru_{n-r} = 0, n \geq r \quad \ldots \text{ (i)}$$

where $p_1, p_2, \ldots, p_r$ are constants. The relation (i) is said to be the *scale of relation* of the recurring series.

Examples.

1. The series $1+2+3+5+8+13+\cdots\cdots$ is a recurring series whose scale of relation is $u_n - u_{n-1} - u_{n-2} = 0, n \geq 2$. The order of the recurring series is 2.

2. The series $1+4+9+16+25+36+\cdots\cdots$ is a recurring series whose scale of relation is $u_n - 3u_{n-1} + 3u_{n-2} - u_{n-3} = 0$. The order of the recurring seties is 3.

The series $u_0 + u_1x + u_2x^2 + \cdots\cdots$ is said to be a recurring series of order r if any $r+1$ *successive coefficients* of the series are connected by the relation

$$u_n + p_1u_{n-1} + p_2u_{n-2} + \cdots + p_ru_{n-r} = 0, n \geq r \quad \ldots \text{ (ii)}$$

where $p_1, p_2, \ldots, p_r$ are constants. The relation (ii) is said to be the *scale of relation* of the recurring series.

Some authors define the scale of relation in a different way.

The series $u_0 + u_1x + u_2x^2 + \cdots\cdots$ is said to be a recurring series of order r if any $r+1$ *successive terms* of the series are connected by the relation

$u_nx^n + p_1x(u_{n-1}x^{n-1}) + p_2x^2(u_{n-2}x^{n-2}) + \cdots + p_rx^r(u_{n-r}x^{n-r}) = 0$, where $n \geq r$ and $p_1, p_2, \ldots, p_r$ are constants. The relation $1 + p_1x + p_2x^2 + \cdots + p_rx^r$ is said to be the *scale of relation* of the recurring series.

We shall accept either of these definitions.

Examples (continued).

3. The series $1+2x+3x^2+4x^3+\cdots\cdots$ is a recurring series of order 2 whose scale of relation is $u_n - 2u_{n-1} + u_{n-2} = 0, n \geq 2$; or, $1 - 2x + x^2$.

4. The series $1+4x+9x^2+16x^3+25x^4+36x^5+\cdots\cdots$ is a recurring series of order 3 whose scale of relation is $1 - 3x + 3x^2 - x^3$,or $u_n - 3u_{n-1} + 3u_{n-2} - u_{n-3} = 0, n \geq 3$.

If the scale of relation and a sufficient number of terms from the beginning of a recurring series be known then the whole series can be constructed.

For example, if $u_0 = 1, u_1 = 3$ and $u_n - 3u_{n-1} - 4u_{n-2} = 0, n \geq 2$ be the scale of relation of the series $u_0 + u_1 + u_2 + \cdots\cdots$ then $u_2, u_3, u_4, \ldots\ldots$ can be determined from the given relation.

Conversely, if a sufficient number of terms from the beginning of a recurring series be given then the scale of relation can be determined.

Theorem 6.4.1. A recurring series of order r is completely determined if the first $2r$ terms are known.

Proof. Let the recurring series of order r be

$$u_0 + u_1x + u_2x^2 + \cdots\cdots$$

whose scale of relation is $1 + p_1x + p_2x^2 + \cdots + p_rx^r$.

Then $u_r + p_1u_{r-1} + p_2u_{r-2} + \cdots + u_0 = 0$

$$u_{r+1} + p_1u_r + p_2u_{r-1} + \cdots + u_1 = 0 \quad \ldots \quad \text{(A)}$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$u_{2r-1} + p_1u_{2r-2} + p_2u_{2r-3} + \cdots + u_{r-1} = 0.$$

In order to determine r unknowns $p_1, p_2, \ldots, p_r$ uniquely, r equations (A) are sufficient. These r equations require $2r$ coefficients $u_0, u_1, \ldots, u_{2r-1}$.

Therefore if $2r$ coefficients $u_0, u_1, \ldots, u_{2r-1}$ be pre-assigned the scale of relation can be fully determined.

Worked Examples.

1. Determine the scale of relation of the recurring series

$$2 + 7x + 25x^2 + 91x^3 + \cdots \cdots$$

Here first four terms are given. Let us assume that the scale of relation is $1 + px + qx^2$.

Then $25 + 7p + 2q = 0$ and $91 + 25p + 7q = 0$.

These determine $p = -7, q = 12$.

Hence the scale of relation is $1 - 7x + 12x^2$.

2. Determine the scale of relation of the recurring series

$$2 + 3 + 5 + 9 + \cdots \cdots$$

Here first four terms are given. Let us assume that the scale of realtion is $u_n + pu_{n-1} + qu_{n-2} = 0$.

Then $5 + 3p + 2q = 0$ and $9 + 5p + 3q = 0$.

These determine $p = -3, q = 2$.

Hence the scale of relation is $u_n - 3u_{n-1} + 2u_{n-2} = 0$.

Theorem 6.4.2. The sum of first n terms of the recurring series $u_0 + u_1x + u_2x^2 + \cdots \cdots$ is a fraction whose denominator is the scale of relation.

Proof. For simplicity we assume the scale of relation as $1 + px + qx^2$.

Whatever be the scale of relation, the method of proof is perfectly general.

$$\begin{aligned} \text{Let } s_n &= u_0 + u_1x + u_2x^2 + \cdots \cdots + u_{n-1}x^{n-1}. \text{Then} \\ pxs_n &= pu_0x + pu_1x^2 + \cdots \cdots + pu_{n-2}x^{n-1} + pu_{n-1}x^n \\ qx^2s_n &= qu_0x^2 + \cdots + qu_{n-3}x^{n-1} + qu_{n-2}x^n + qu_{n-1}x^{n+1} \end{aligned}$$

Therefore $(1 + px + qx^2)s_n = u_0 + (u_1 + pu_0)x + (pu_{n-1} + qu_{n-2})x^n + qu_{n-1}x^{n+1}$, since $u_n + pu_{n-2} + qu_{n-3} = 0, n \geq 2$.

$$\text{or, } s_n = \frac{u_0 + (u_1 + pu_0)x + (pu_{n-1} + qu_{n-2})x^n + qu_{n-1}x^{n+1}}{1 + px + qx^2}.$$

Note. The sum s_n can be expressed as

$$s_n = \frac{u_0 + (u_1 + pu_0)x}{1 + px + qx^2} + \frac{(pu_{n-1} + qu_{n-2})x^n + qu_{n-1}x^{n+1}}{1 + px + qx^2}.$$

If the second term tends to zero as n increases indefinitely, then the series $u_0 + u_1x + u_2x^2 + \cdots \cdots$ is convergent and in that case the sum is $\frac{u_0 + (u_1 + pu_0)x}{1 + px + qx^2}$.

If we develop the fraction $\frac{u_0 + (u_1 + pu_0)x}{1 + px + qx^2}$ is ascending powers of x, then we shall obtain as many terms of the series as we please but it will be the *sum* of the infinite series $u_0 + u_1x + u_2x^2 + \cdots \cdots$ only if the remainder $\frac{(pu_{n-1} + qu_{n-2})x^n + qu_{n-1}x^{n+1}}{1 + px + qx^2}$ tends to zero as n increases indefinitely.

For this reason the fraction $\frac{u_0 + (u_1 + pu_0)x}{1 + px + qx^2}$ is said to be the *generating function* of the series $u_0 + u_1x + u_2x^2 + \cdots \cdots$.

To illustrate, let us consider the series $1 + x + x^2 + x^3 + \cdots \cdots$.

This is a recuring series of order 1. The scale of relation is $1 - x$.

Let $s_n = 1 + x + x^2 + \cdots + x^{n-1}$. Then

$$\begin{aligned} -xs_n = \quad & -x - x^2 - \cdots - x^{n-1} - x^n \end{aligned}$$

Therefore $(1 - x)s_n = 1 - x^n$,

or, $s_n = \frac{1 - x^n}{1 - x} = \frac{1}{1 - x} - \frac{x^n}{1 - x}$.

The generating function is $\frac{1}{1 - x}$. Expanding in ascending powers of x,

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1 - x}.$$

Thus we obtain as many terms of the series as we please, but $\frac{1}{1 - x}$ will be equivalent to $1 + x + x^2 + \cdots$ if and only if $\frac{x^n}{1 - x}$ tends to zero as n increases indefinitely. This happens only when $|x| < 1$, in which case the series is convergent and the sum of the infinite series is $\frac{1}{1 - x}$.

Worked Examples (continued).

3. Find the generating function, the general term and the sum to n terms of the series $2 + 3x + 5x^2 + 9x^3 + \cdots \cdots$

Let the scale of relation be $1 + px + qx^2$.

Then $5 + 3p + 2q = 0, \quad 9 + 5p + 3q = 0$.

These determine $p = -3, q = 2$.

Hence the scale of relation is $1 - 3x + 2x^2$.

Let s denote the sum of the series, when it is convergent.

$$\begin{aligned} \text{Then } s &= 2 + 3x + 5x^2 + 9x^3 + \cdots \cdots \\ -3xs &= \quad -6x - 9x^2 - 15x^3 - \cdots \cdots \\ 2x^2s &= \quad\quad 4x^2 + 6x^3 + \cdots \cdots \end{aligned}$$

Adding, we have $s(1 - 3x + 2x^2) = 2 - 3x$, or, $s = \frac{2 - 3x}{1 - 3x + 2x^2}$.

The generating function is $\frac{2 - 3x}{1 - 3x + 2x^2} = \frac{1}{1 - x} + \frac{1}{1 - 2x}$.

The coefficient of x^r in $\frac{1}{1 - x}$ is 1^r, and the coefficient of x^r in $\frac{1}{1 - 2x}$ is 2^r.

Hence the $(r + 1)$th term of the series is $(1 + 2^r)x^r$.

$$\begin{aligned}\text{The sum to } n \text{ terms} &= \sum_{r=0}^{n-1}(1+2^r)x^r \\ &= (1+x+x^2+\cdots+x^{n-1}) \\ &\quad +(1+2x+2^2x^2+\cdots+2^{n-1}x^{n-1}) \\ &= \tfrac{1-x^n}{1-x}+\tfrac{1-2^nx^n}{1-2x}.\end{aligned}$$

4. Find the general term and the sum of first n terms of the series $2+7+25+91+\cdots\cdots$

We shall find the general term and the sum of first n terms of the recurring series $2+7x+25x^2+91x^3+\cdots\cdots$ and put $x=1$ in the results.

The scale of relation can be obtained as $1-7x+12x^2$.

Let s be the sum of the infinite series when it is convergent.

$$\begin{aligned}s &= 2+7x+25x^2+91x^3+\cdots\cdots \\ -7xs &= \quad -14x-49x^2-175x^3-\cdots\cdots \\ 12x^2s &= \qquad 24x^2+84x^3+\cdots\cdots\end{aligned}$$

Then $s(1-7x+12x^2)=2-7x$

or, $s=\frac{2-7x}{1-7x+12x^2}=\frac{1}{1-3x}+\frac{1}{1-4x}$.

The coefficient of x^r in the expansion of $\frac{1}{1-3x}$ is 3^r and the coefficient of x^r in the expansion of $\frac{1}{1-4x}$ is 4^r.

Hence the $(r+1)$th term of the series is $(3^r+4^r)x^r$.

$$\begin{aligned}\text{The sum to } n \text{ terms} &= \sum_{r=0}^{n-1}(3^r+4^r)x^r \\ &= (1+3x+3^2x^2+\cdots+3^{n-1}x^{n-1}) \\ &\quad +(1+4x+4^2x^2+\cdots+4^{n-1}x^{n-1}) \\ &= \tfrac{1-3^nx^n}{1-3x}+\tfrac{1-4^nx^n}{1-4x}.\end{aligned}$$

Hence the $(r+1)$th term of the given series is 3^r+4^r and the sum of first n terms of the given series is $\frac{1}{2}(3^n-1)+\frac{1}{3}(4^n-1)$.

Theorem 6.4.3. If u_n is a polynomial in n of degree p then the series $u_0+u_1x+u_2x^2+\cdots\cdots$ is a recurring series of which the scale of relation is $(1-x)^{p+1}$.

Proof. Let s be the sum of the series when it is convergent.

$$\begin{aligned}s &= u_0+u_1x+u_2x^2+\cdots+u_nx^n+\cdots \\ s(1-x) &= u_0+(u_1-u_0)x+\cdots+(u_n-u_{n-1})x^n+\cdots \\ &= u_0+v_1x+v_2x^2+\cdots+v_nx^n+\cdots, \text{ where}\end{aligned}$$

$v_n=u_n-u_{n_1}$ and so v_n is a polynomial in n of degree $p-1$.

$$\begin{aligned}s(1-x)^2 &= u_0+(v_1-u_0)x+\cdots+(v_n-v_{n-1})x^n+\cdots \\ &= u_0+(v_1-u_0)x+w_2x^2+\cdots+w_nx^n+\cdots, \text{ where}\end{aligned}$$

$w_n=v_n-v_{n-1}, n\geq 2$ and so w_n is a polynomial of degree $p-2$.

Proceeding in a similar manner, we have

$s(1-x)^p=f(x)+k(x^p+x^{p+1}+\cdots)$, where $f(x)$ is the sum of first p terms and $k(=\Delta^p u_n)$ is a constant since u_n is a polynomial of degree p.

Therefore $s(1-x)^p=f(x)+\frac{kx^p}{1-x}$

or, $s=\frac{(1-x)f(x)+kx^p}{(1-x)^{p+1}}$, and this is the generating function of the series. Therefore the series is a recurring whose scale of relation is $(1-x)^{p+1}$.

Worked Example (continued).

5. Find the generating function of the series

$$2+5x+10x^2+17x^3+26x^4+\cdots\cdots$$

The sequence of coefficients and the succesive orders of differences are

2		5		10		17		26 ...
	3		5		7		9	... ...
		2		2		2	...	... ...

Since the second order of differences is a sequence of equal elements, u_n is a polynomial of degree 2. Therefore the series is a recurring series and the scale of relation is $(1-x)^3$.

Let s be the sum of the series when it is convergent.

$$\begin{aligned}\text{Then } s &= 2+5x+10x^2+17x^3+26x^4+\cdots\cdots \\ -3xs &= \quad -6x-15x^2-30x^3-51x^4-\cdots\cdots \\ 3x^2s &= \qquad 6x^2+15x^3+30x^4+\cdots\cdots \\ -x^3s &= \qquad\quad -2x^3-5x^4-\cdots\cdots\end{aligned}$$

Adding, we have $(1-x)^3s=2-x+x^2$, or $s=\frac{2-x+x^2}{(1-x)^3}$.

Therefore the generating function is $\frac{2-x+x^2}{(1-x)^3}$.

Theorem 6.4.4. Let $u_0+u_1+u_2+\cdots$ be a recurring series of real numbers whose scale of relation is $u_n+pu_{n-1}+qu_{n-2}=0$ where p,q are constants.

If α,β be the roots of the equation $x^2+px+q=0$ then

(i) if $\alpha\neq\beta, u_n=A\alpha^n+B\beta^n$ where A,B are arbitrary constants

(ii) if $\alpha = \beta, u_n = (A + nB)\alpha^n$ where A, B are arbitrary constants.

Proof. Let us consider the series $u_0 + u_1x + u_2x^2 + \dots$

Assuming convergence of the series, let s be the sum of the series.

$$s = u_0 + u_1x + u_2x^2 + \cdots$$
$$px s = \quad pu_0x + pu_1x^2 + \cdots$$
$$qx^2s = \quad qu_0x^2 + \cdots$$

We have $(1 + px + qx^2)s = u_0 + (u_1 + pu_0)x$.

or, $s = \frac{u_0+(u_1+pu_0)x}{1+px+qx^2} = \frac{u_0+(u_1+pu_0)x}{(1-\alpha x)(1-\beta x)}$.

(i) Since $\alpha \neq \beta$, we have

$s = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$, where A, B are constants

$= A(1+\alpha x+\alpha^2x^2+\cdots+\alpha^nx^n+\cdots)+B(1+\beta x+\beta^2x^2+\cdots+\beta^nx^n+\cdots)$,
where $|x| < \min\{|\alpha|, |\beta|\}$.

Therefore u_n = coefficient of x^n in R.H.S.
$= A\alpha^n + B\beta^n$.

(ii) Since $\alpha = \beta$, we have

$s = \frac{A'}{1-\alpha x} + \frac{B'}{(1-\alpha x)^2}$, where A', B' are constants

$= A'(1 + \alpha x + \alpha^2x^2 + \cdots + \alpha^nx^n + \cdots) + B'(1 + 2\alpha x + 3\alpha^2x^2 + \cdots + (n+1)\alpha^nx^n + \cdots)$, where $|x| < |\alpha|$.

Therefore u_n = coefficient of x^n in R.H.S.
$= A'\alpha^n + B'(n+1)\alpha^n$
$= (A + nB)\alpha^n$, where A, B are constants.

Theorem 6.4.5. Let $u_0 + u_1 + u_2 + \cdots$ be a recurring series of real numbers whose scale of relation is $u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} = 0$ where p, q, r are constants.

If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$ then

(i) if no two of α, β, γ are equal, $u_n = A\alpha^n + B\beta^n + c\gamma^n$, where A, B, C are constants;

(ii) if $\alpha = \beta \neq \gamma, u_n = (A+nB)\alpha^n + C\gamma^n$, where A, B, C are constants;

(iii) if $\alpha = \beta = \gamma, u_n = (A + nB + n^2C)\alpha^n$, where A, B, C are constants.

Proof. Left to the reader.

Worked Examples (continued).

11. Find the nth term of the recurring series

$$1 + 2 + 5 + 14 + \cdots$$

Let the series be $u_1 + u_2 + u_3 + \cdots$ and let the scale of relation be $u_n + pu_{n-1} + qu_{n-2} = 0$.

Then $5 + 2p + q = 0$, $14 + 5p + 2q = 0$, giving $p = -4, q = 3$.

The roots of the equation $x^2 - 4x + 3 = 0$ are 1, 3.

Hence $u_n = A.1^n + B.3^n$ where A, B are constants.

We have $A + 3B = 1, A + 9B = 2$, giving $A = \frac{1}{2}, B = \frac{1}{6}$.

Hence $u_n = \frac{1}{2}(1 + 3^{n-1})$.

12. Find the nth term of the recurring series

$$4 + 7 + 11 + 17 + 27 + 45 + \cdots.$$

Let the series be $u_1 + u_2 + u_3 + \cdots$ and let the scale of relation be $u_n + pu_{n-1} + q_{n-2} + ru_{n-3} = 0$.

Then $17 + 11p + 7q + 4r = 0$, $27 + 17p + 11q + 7r = 0$, $45 + 27p + 17q + 11r = 0$, giving $p = -4, q = 5, r = -2$.

The roots of the equation $x^3 - 4x^2 + 5x - 2 = 0$ are $1, 1, 2$.

Hence $u_n = (A + nB)1^n + C.2^n$ where A, B, C are constants.

Since $u_1 = 4, u_2 = 7, u_3 = 11$, we have $A + B + 2C = 4$, $A + 2B + 4C = 7$, $A + 3B + 8C = 11$, giving $A = 1, B = 2, C = \frac{1}{2}$.

Hence $u_n = 1 + 2n + 2^{n-1}$.

Exercises 6B

1. Find the nth term and the sum of n terms of the series:

(i) $5 + 9 + 15 + 23 + 33 + \cdots$ $\cdots$

(ii) $2 + 10 + 30 + 68 + 130 + \cdots$ $\cdots$

(iii) $2.6 + 3.14 + 4.24 + 5.36 + \cdots$ $\cdots$

(iv) $9 + 17 + 37 + 93 + 257 + \cdots$ $\cdots$

(v) $3 + 5 + 10 + 22 + 49 + 107 + \cdots$ $\cdots$

(vi) $3 + 6 + 13 + 32 + 87 + 250 + \cdots$ $\cdots$

(vii) $\frac{3}{1.2}\cdot\frac{1}{4} + \frac{4}{2.3}\cdot\frac{1}{8} + \frac{5}{3.4}\cdot\frac{1}{16} + \frac{6}{4.5}\cdot\frac{1}{32} + \cdots$ $\cdots$

(viii) $3.1 + 5.3 + 7.3^2 + 9.3^3 + 11.3^4 + \cdots$ $\cdots$

2. Find the general term and the sum of n terms of the series:

(i) $2 + 5 + 13 + 35 + \cdots$ $\cdots$

(ii) $2 + 6 + 20 + 72 + \cdots$ $\cdots$

(iii) $4+12+32+80+\cdots\cdots$

(iv) $3+7+9+19+33+67+\cdots\cdots$

6.5. $C+iS$ method of summation.

We now discuss a method of finding the sum of two real series

$a_0+a_1\cos\theta+a_2\cos 2\theta+\cdots\cdots$ (i)

$a_1\sin\theta+a_2\sin 2\theta+\cdots\cdots$ (ii)

where θ is real and $a_0, a_1, a_2, \cdots$ are constants.

Let the complex series

$a_0+a_1e^{i\theta}+a_2e^{2i\theta}+\cdots\cdots$ (iii)

be convergent for some θ and $f(\theta)$ be the sum. Let $f(\theta)$ be expressed as $A+iB$ where A, B are real. Then

$A+iB=a_0+a_1(\cos\theta+i\sin\theta)+a_2(\cos 2\theta+i\sin 2\theta)+\ldots$

This implies the convergence of the series (i) and (ii) and

$A=a_0+a_1\cos\theta+a_2\cos 2\theta+\cdots\cdots$

$B=a_1\sin\theta+a_2\sin 2\theta+\cdots\cdots$

Therefore in order to find the sum of either of the series (i) and (ii) we try to find the sum of the complex series (iii) in its region of convergence and then express the sum $f(\theta)$ in the form $C+iS$ where C and S are real.

The sums of the series (i) and (ii) for the corresponding θ are given by C and S respectively.

Examples.

1. Find the sum of the series

$\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\cdots\cdots \quad (-\pi<\theta<\pi)$.

Assuming convergence of the series

$\cos\theta-\frac{1}{2}\cos 2\theta+\frac{1}{3}\cos 3\theta-\cdots\cdots$

and $\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\cdots\cdots$

with C and S as their sum respectively, we have

$$\begin{aligned} C+iS &= e^{i\theta}-\tfrac{1}{2}e^{2i\theta}+\tfrac{1}{3}e^{3i\theta}-\cdots\cdots \\ &= \log_e(1+e^{i\theta})=\log_e(1+\cos\theta+i\sin\theta). \end{aligned}$$

Let $1+\cos\theta=r\cos\phi, \sin\theta=r\sin\phi$.

Then $r=2\cos\frac{\theta}{2}$ and $\phi=\frac{\theta}{2}$.

Therefore $C+iS=\log(1+\cos\theta+i\sin\theta)=\log(2\cos\frac{\theta}{2})+i\frac{\theta}{2}$.

This gives $S=\frac{\theta}{2}$ and $C=\log(2\cos\frac{\theta}{2})$.

Hence $\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\cdots=\frac{\theta}{2}$

and $\cos\theta-\frac{1}{2}\cos 2\theta+\frac{1}{3}\cos 3\theta-\cdots=\log(2\cos\frac{\theta}{2})$, when $-\pi<\theta<\pi$.

Note. The restriction on θ is necessary for the convergence of the series. The series $z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots\cdots$ converges to $\log_e(1+z)$ when $|z|\leq 1$ but $z\neq -1$.

2. Find the sum of the series

$\cos\theta-\frac{1}{3}\cos 3\theta+\frac{1}{5}5\theta-\cdots\cdots \quad (0<\theta<\frac{\pi}{2})$

Assuming convergence of the series

$\cos\theta-\frac{1}{3}\cos 3\theta+\frac{1}{5}\cos 5\theta-\cdots\cdots$

and $\sin\theta-\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta-\cdots\cdots$

with c and s as their sum respectively, we have

$$\begin{aligned} c+is &= e^{i\theta}-\tfrac{1}{3}e^{3i\theta}+\tfrac{1}{5}e^{5i\theta}-\cdots\cdots \\ &= \tan^{-1}(e^{i\theta})=\tan^{-1}(\cos\theta+i\sin\theta). \end{aligned}$$

Therefore $\frac{\sin(c+is)}{\cos(c+is)}=\cos\theta+i\sin\theta$

or, $\frac{\cos(c+is)+i\sin(c+is)}{\cos(c+is)-i\sin(c+is)}=\frac{(1-\sin\theta)+i\cos\theta}{(1+\sin\theta)-i\cos\theta}$

or, $\frac{\exp i(c+is)}{\exp -i(c+is)}=\frac{\{(1-\sin\theta)+i\cos\theta\}\{(1+\sin\theta)+i\cos\theta\}}{(1+\sin\theta)^2+\cos^2\theta}$

or, $e^{-2s}(\cos 2c+i\sin 2c)=\frac{2i\cos\theta}{2(1+\sin\theta)}$

Therefore $e^{-2s}\cos 2c=0, e^{-2s}\sin 2c=\frac{\cos\theta}{1+\sin\theta}$.

This implies $e^{2s}=\frac{1+\sin\theta}{\cos\theta}$ and $2c=\frac{\pi}{2}$.

Therefore $s=\frac{1}{2}\log(\sec\theta+\tan\theta)$ and $c=\frac{\pi}{4}$.

Hence $\cos\theta-\frac{1}{3}\cos 3\theta+\cdots=\frac{\pi}{4}$, when $0<\theta<\frac{\pi}{2}$

Note. The series $z-\frac{z^3}{3}+\frac{z^5}{5}-\cdots$ converges to $\tan^{-1}z$ when $|z|\leq 1$ but $z\neq\pm i$.

Exercises 6C

1. Find the sum of the following series:

(i) $1+\cos\theta+\frac{\cos 2\theta}{2!}+\frac{\cos 3\theta}{3!}+\cdots\cdots$

(ii) $1+c\cos\theta+c^2\cos 2\theta+c^3\cos 3\theta+\cdots\cdots \quad (0<c<1)$

(iii) $\cos\theta+x\cos 2\theta+x^2\cos 3\theta+\cdots\cdots \quad (0<x<1)$

(iv) $1+x\cos\theta+\frac{x^2}{2!}\cos 2\theta+\frac{x^3}{3!}\cos 3\theta+\cdots\cdots$

(v) $\cos 2\theta - \frac{1}{2}\cos 4\theta + \frac{1}{3}\cos 6\theta - \cdots\cdots \quad (-\frac{\pi}{2} < \theta < \frac{\pi}{2})$

(vi) $\cos\theta \sin\theta + \frac{\cos^2\theta}{2}.\sin 2\theta + \frac{\cos^3\theta}{3}\sin 3\theta + \cdots\cdots \quad (0 < \theta < \pi)$

(vii) $\sin 2\theta + \frac{1}{2}\sin 4\theta + \frac{1}{3}\sin 6\theta + \cdots\cdots \quad (0 < \theta < \pi)$

(viii) $\cos^2\theta - \frac{1}{2}\cos^2 2\theta + \frac{1}{3}\cos^2 3\theta - \cdots\cdots \quad (-\frac{\pi}{2} < \theta < \frac{\pi}{2})$

(ix) $\cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cos^3\theta\cos 3\theta + \cdots\cdots \quad (\theta \neq n\pi)$

(x) $\sin^2\theta - \frac{1}{2}\sin^2 2\theta + \frac{1}{3}\sin^2 3\theta - \cdots\cdots \quad (-\frac{\pi}{2} < \theta < \frac{\pi}{2})$

2. If a, b, c be three sides of a triangle and $a > b$, prove that

(i) $\frac{b}{a}\cos C + \frac{b^2}{2a^2}\cos 2C + \frac{b^3}{3a^3}\cos 3C + \cdots \quad \cdots = \log\frac{a}{c}$

(ii) $\frac{b}{a}\sin C + \frac{b^2}{2a^2}\sin 2C + \frac{b^3}{3a^3}\sin 3C + \cdots \quad \cdots = B$

(iii) $1 + \frac{b}{a}\cos C + \frac{b^2}{a^2}\cos 2C + \frac{b^3}{a^3}\cos 3C + \cdots \quad \cdots = \frac{a}{c}\cos B$

(iv) $\frac{b}{a}\sin C + \frac{b^2}{a^2}\sin 2C + \frac{b^3}{a^3}\sin 3C + \cdots \quad \cdots = \frac{a}{c}\sin B.$

7. SIMPLE CONTINUED FRACTION

7.1. Continued fraction.

An expression of the form $a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}$

where a's and b's are all integers with $b_i \neq 0$ for $i \geq 2, a_i > 0$ for $i > 1, a_1 \geq 0$, is called a *continued fraction.*

The continued fraction is also denoted by $a_1 + \frac{b_2}{a_2+}\frac{b_3}{a_3+}\cdots$

$a_1, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \cdots$ are called *elements* of the continued fraction.

If the number of elements be finite, the continued fraction is said to be a *finite continued fraction.*

If the number of elements be infinite, the continued fraction is said to be an *infinite continued fraction.*

The value of the fraction obtained by stopping at some stage is called a *convergent* of the continued fraction.

For the continued fraction $a_1 + \frac{b_2}{a_2+}\frac{b_3}{a_3+}\cdots$
the first convergent $= a_1$,
the second convergent $= a_1 + \frac{b_2}{a_2}$,
the third convergent $= a_1 + \frac{b_2}{a_2+}\frac{b_3}{a_3}$,
... ...

As the number of convergents is finite in a finite continued fraction, the value of the last convergent gives the *value* of a finite continued fraction.

In an infinite continued fraction, the number of convergents is clearly infinite. If $u_1, u_2, u_3, \cdots$ be the successive convergents of the continued fraction, the continued fraction is said to be *convergent* if the sequence $\{u_n\}$ be convergent. In this case $\lim u_n$ is said to be the *value* of the continued fraction.

If however, the sequence $\{u_n\}$ be not convergent, the continued fraction is said to be *divergent.*

For example, the continued fraction $1 + \frac{1}{2+}\frac{1}{2+}\cdots$ is convergent, but the continued fraction $\frac{1}{1+}\frac{-1}{2+}\frac{-1}{2+}\cdots$ is divergent.

7.2. Simple continued fraction.

The continued fraction $a_1 + \frac{b_2}{a_2+}\frac{b_3}{a_3+}\cdots$ in which $b_i = 1$ for all $i \geq 2$ and $a_i > 0$ for all $i > 1, a_1 \geq 0$ is said to be a *simple continued fraction*. Thus the general form of a simple continued fraction is

$$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$$

a_i is called the ith *quotient*.

$a_1, a_1+\frac{1}{a_2}, a_1+\frac{1}{a_2+}\frac{1}{a_3+}\cdots$ are called respectively the first convergent, the second convergent, the third convergent, ...

Examples.

1. For the simple continued fraction $1 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3}$,

   the first convergent $= 1$,
   the second convergent $= 1 + \frac{1}{2} = \frac{3}{2}$,
   the third convergent $= 1 + \frac{1}{2+}\frac{1}{1} = \frac{4}{3}, \cdots, \cdots$
   the fourth convergent $= 1 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3} = \frac{15}{11}$.

   Here the value of the continued fraction is $\frac{15}{11}$.

2. For the simple continued fraction $\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3}$,

   the first convergent $= 0$,
   the second convergent $= \frac{1}{1}$,
   the third convergent $= \frac{1}{1+}\frac{1}{2+} = \frac{2}{3}$,
   the fourth convergent $= \frac{1}{1+}\frac{1}{2+}\frac{1}{1+} = \frac{3}{4}$,
   the fifth convergent $= \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3} = \frac{11}{15}$.

   Here the value of the continued fraction is $\frac{11}{15}$.

Theorem 7.2.1. If $u_n(= \frac{p_n}{q_n})$ be the nth convergent of the simple continued fraction $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$ then the recurrence relations

$p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ hold for all $n \geq 3$.

Proof. $u_1 = a_1$. Therefore $p_1 = a_1, q_1 = 1$.

$u_2 = a_1 + \frac{1}{a_2} = \frac{a_2 a_1 + 1}{a_2}$. Therefore $p_2 = a_2 a_1 + 1, q_2 = a_2$.

$u_3 = a_1 + \frac{1}{a_2+}\frac{1}{a_3} = a_1 + \frac{a_3}{a_3 a_2 + 1} = \frac{a_3(a_2 a_1 + 1) + a_1}{a_3 a_2 + 1}$.

Therefore $p_3 = a_3(a_2 a_1 + 1) + a_1 = a_3 p_2 + p_1$, $q_3 = a_3 a_2 + 1 = a_3 q_2 + q_1$.
This shows that the relations hold for $n = 3$.

Let us assume that the relations hold for $n = m$ where m is a natural number ≥ 3.

Then $u_m = \frac{p_m}{q_m} = \frac{a_m p_{m-1} + p_{m-2}}{a_m q_{m-1} + q_{m-2}}$.

$u_{m+1} = a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots\frac{1}{a_{m+1}} = a_1 + \frac{1}{a_2+}\cdots\frac{1}{(a_m + \frac{1}{a_{m+1}})}$.

We observe that u_{m+1} is obtained from u_m if a_m be replaced by $a_m + \frac{1}{a_{m+1}}$.

Then $u_{m+1} = \frac{p_{m+1}}{q_{m+1}} = \frac{(a_m + \frac{1}{a_{m+1}})p_{m-1} + p_{m-2}}{(a_m + \frac{1}{a_{m+1}})q_{m-1} + q_{m-2}}$

$= \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}}$

$= \frac{a_{m+1}p_m + p_{m-1}}{a_{m+1}q_m + q_{m-1}}$.

Therefore $p_{m+1} = a_{m+1}p_m + p_{m-1}$, $q_{m+1} = a_{m+1}q_m + q_{m-1}$.

This shows that the relations hold for $n = m+1$ if they hold for $n = m$.

By the principle of induction, the relations hold for all $n \geq 3$.

Note. If we write $p_0 = 1, q_0 = 0$ then p_2, q_2 can be expressed as

$p_2 = a_2 p_1 + p_0$, $q_2 = a_2 q_1 + q_0$.

This shows that the recurrence relations

$p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$

hold for all $n \geq 2$ with the assumed initial values $p_0 = 1, q_0 = 0$.

If a simple continued fraction be finite then there is a finite number of quotients and the last convergent gives the value of the continued fraction.

Since each convergent u_n is of the form $\frac{p_n}{q_n}$ where p_n, q_n are positive integers, each convergent is a rational number and therefore the value of a finite simple continued fraction is a positive rational number.

We now show that every positive rational number can be expressed as a finite simple continued fraction.

Theorem 7.2.2. Every positive rational number r can be expressed as a finite simple continued fraction.

Proof. Let $r = \frac{p}{q}$ be a positive rational number, where p, q are integers and $p > 0, q > 0$.

Case 1. $p > q$. Let us divide p by q. By division algorithm, there exist integers a_1 and r_1 such that $p = a_1 q + r_1$ where $a_1 \geq 1, 0 \leq r_1 < q$.

Then $\frac{p}{q} = a_1 + \frac{r_1}{q}$.

If $r_1 = 0$ then $\frac{p}{q} = a_1$ and it is a simple continued fraction having only one quotient.

If $0 < r_1 < q$, $\frac{p}{q} = a_1 + \frac{r_1}{q} = a_1 + \frac{1}{\frac{q}{r_1}}$.

Let us divide q by r_1. By division algorithm, there exist integers a_2 and r_2 such that $q = a_2 r_1 + r_2$ where $a_2 \geq 1, 0 \leq r_2 < r_1$.

If $r_2 = 0$, then $\frac{q}{r_1} = a_2$ and $\frac{p}{q} = a_1 + \frac{1}{a_2}$, a simple continued fraction having only two quotients.

If $0 < r_2 < r_1$ then $\frac{q}{r_1} = a_2 + \frac{r_2}{r_1} = a_2 + \frac{1}{\frac{r_1}{r_2}}$.

Therefore $\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{r_1}{r_2}}$.

Let us divide r_1 by r_2.

Proceeding in this way we obtain successive remainders $r_1, r_2, r_3, \ldots$ such that $q > r_1 > r_2 > r_3 > \cdots$

Since q is a positive integer and r_i's are all integers ≥ 0, we shall ultimately arrive at a stage where some $r_n = 0$. In this case we have $r_{n-2} = a_n r_{n-1} + 0$ and therefore

$$\frac{p}{q} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n}.$$

Thus the rational number $\frac{p}{q}$ is expressed as a simple continued fraction $a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_{n-1} + \frac{1}{a_n}}$ which is finite.

Case 2. $p < q$. Then $\frac{p}{q} = \frac{1}{\frac{q}{p}}$ and $q > p$.

By case 1, $\frac{q}{p} = b_1 + \frac{1}{b_2 +} \cdots \frac{1}{b_n}$ where b_i are integers ≥ 1.

Therefore $\frac{p}{q} = \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots \frac{1}{b_n}$ and it is a finite simple continued fraction.

Note 1. It can be so arranged that the simple continued fraction representing $\frac{p}{q}$ may have an even (or an odd) number of quotients.

Let $\frac{p}{q} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n}$, having n quotients.

If $a_n = 1$, $\frac{p}{q}$ can be expressed as $a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_{n-1} + 1}$, a finite simple continued fraction having $n - 1$ quotients.

If $a_n > 1$, $\frac{p}{q}$ can be expressed as $a_1 + \frac{1}{a_2 +} \cdots \frac{1}{(a_n - 1) +} \frac{1}{1}$, a finite simple continued fraction having $n + 1$ quotients.

It follows that if n is odd, then $\frac{p}{q}$ can also be expressed as a simple continued fraction having an even number of quotients; and if n is even, then $\frac{p}{q}$ can also be expressed as a simple continued fraction having an odd number of quotients.

Note 2. The quotients $a_1, a_2, \cdots, a_n$ in the simple continued fraction representing $\frac{p}{q}$ can be obtained as successive remainders in the process of Euclidean algorithm in finding the g.c.d. of p and q.

Worked Example.

1. Express $\frac{38}{15}$ as a finite simple continued fraction having

(i) an even number of quotients, (ii) an odd number of quotients.

(i) $\frac{38}{15} = 2 + \frac{8}{15} = 2 + \frac{1}{\frac{15}{8}}$, $\quad \frac{15}{8} = 1 + \frac{7}{8} = 1 + \frac{1}{\frac{8}{7}}$, $\quad \frac{8}{7} = 1 + \frac{1}{7}$.

Therefore $\frac{38}{15} = 2 + \frac{1}{1 +} \frac{1}{1 +} \frac{1}{7}$, a simple continued fraction having an even number of quotients.

(ii) Again $\frac{38}{15} = 2 + \frac{1}{1 +} \frac{1}{1 +} \frac{1}{6 +} \frac{1}{1}$, a simple continued fraction having an odd number of quotients.

Theorem 7.2.3. A positive irrational number can be expressed as an infinite simple continued fraction uniquely.

Proof. Let α be a positive irrational number. Then there exists an integer $a_1 \geq 0$ such that $a_1 < \alpha < a_1 + 1$. Clearly, $a_1 = [\alpha]$, the integral part of α.

$\alpha = a_1 + (\alpha - a_1) = a_1 + \alpha_1$, where α_1 is an irrational number and $0 < \alpha_1 < 1$. Therefore $\alpha = a_1 + \frac{1}{\frac{1}{\alpha_1}}$.

$\frac{1}{\alpha_1} > 1$. There exists an integer $a_2 \geq 1$ such that $\frac{1}{\alpha_1} = a_2 + \alpha_2$ where α_2 is an irrational number and $0 < \alpha_2 < 1$.

Therefore $\alpha = a_1 + \frac{1}{a_2 + \alpha_2} = a_1 + \frac{1}{a_2 + \frac{1}{\frac{1}{\alpha_2}}}$.

$\frac{1}{\alpha_2} > 1$. The process can be continued indefinitely because at each step we obtain an irrational number α_i such that $0 < \alpha_i < 1$ and a positive integer $a_i \geq 1$ such that $a_i < \frac{1}{\alpha_i} < a_i + 1$.

Thus α is expressed as an infinite simple continued fraction

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots$$

We now prove that the representation is unique.

Let α be expressed as another infinite simple continued fraction

$$b_1 + \frac{1}{b_2 +} \frac{1}{b_3 +} \cdots$$

Then $a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} + \cdots = b_1 + \frac{1}{b_2 +} \frac{1}{b_3 +} \cdots$

Equating the integral parts of both sides, we have $a_1 = b_1$ and $\frac{1}{a_2 +} \frac{1}{a_3 +} \cdots = \frac{1}{b_2 +} \frac{1}{b_3 +} \cdots$ Therefore $a_2 + \frac{1}{a_3 +} \cdots = b_2 + \frac{1}{b_3 +} \cdots$

Equating the integral parts of both sides, we have $a_2 = b_2$ and $\frac{1}{a_3 +} \frac{1}{a_4 +} \cdots = \frac{1}{b_3 +} \frac{1}{b_4 +} \cdots$ Therefore $a_3 + \frac{1}{a_4 +} \cdots = b_3 + \frac{1}{b_4 +} \cdots$

Proceeding similarly, we have $a_3 = b_3, \cdots, a_n = b_n$ for all $n \in \mathbf{N}$.

Thus the representation of α as an infinite simple continued fraction is unique.

Note. Before coming to the converse problem whether an infinite simple continued fraction represents an irrational number, we must first ascertain whether an infinite simple continued fraction has a value at all, i.e., whether the sequence of convergents of an infinite simple continued fraction converges to a limit. This will be discussed in a subsequent theorem.

Worked Example (continued).

2. Express $\sqrt{5}$ as a simple continued fraction.

$2 < \sqrt{5} < 3$. $\sqrt{5}$ can be expressed as $2 + (\sqrt{5} - 2)$.

$\sqrt{5} - 2$ is an irrational number and $0 < \sqrt{5} - 2 < 1$.

$\sqrt{5} = 2 + (\sqrt{5} - 2) = 2 + \frac{1}{\frac{1}{\sqrt{5}-2}} = 2 + \frac{1}{\sqrt{5}+2}$. $4 < \sqrt{5} + 2 < 5$.

Therefore $\sqrt{5} + 2 = 4 + (\sqrt{5} - 2) = 4 + \frac{1}{\frac{1}{\sqrt{5}-2}} = 4 + \frac{1}{\sqrt{5}+2}$.

Hence $\sqrt{5} = 2 + \frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\cdots$

7.3. Properties of the convergents.

7.3.1. If $\frac{p_r}{q_r}$ be the rth convergent of the simple continued fraction

$$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$$

then the sequences $\{p_n\}$ and $\{q_n\}_2^\infty$ are strictly increasing sequences of positive integers.

Proof. p_n and q_n are defined by $p_1 = a_1, q_1 = 1, p_2 = a_2a_1 + 1, q_2 = a_2$ and for $n > 2$,

$p_n = a_np_{n-1} + p_{n-2}, q_n = a_nq_{n-1} + q_{n-2}$.

Since each a_i is a positive integer for $i \geq 2$, p_n is a positive integer for $n \geq 2$ and q_n is a positive integer for all $n \in \mathbb{N}$.

We have $p_2 > p_1$ and for $n \geq 2$,

$p_{n+1} - p_n = (a_{n+1} - 1)p_n + p_{n-1} \geq p_{n-1} > 0$.

Hence $p_{n+1} > p_n$ for all $n \in \mathbb{N}$.

For $n \geq 2$, $q_{n+1} - q_n = (a_{n+1} - 1)q_n + q_{n-1} \geq q_{n-1} > 0$.

Hence $q_{n+1} > q_n$ for all $n \geq 2$.

Therefore the sequences $\{p_n\}$ and $\{q_n\}_2^\infty$ are strictly increasing sequences of positive integers.

Note. If the simple continued fraction be infinite, both the sequences $\{p_n\}$ and $\{q_n\}$ diverge to infinity.

7.3.2. If $\frac{p_r}{q_r}$ be the rth convergent of the simple continued fraction

$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$,

then $p_nq_{n-1} - p_{n-1}q_n = (-1)^n$ for all $n \geq 1$.

Proof. p_n and q_n are defined by $p_0 = 1, q_0 = 0, p_1 = a_1, q_1 = 1$ and for $n \geq 2$, $p_n = a_np_{n-1} + p_{n-2}, q_n = a_nq_{n-1} + q_{n-2}$.

Since $p_1q_0 - p_0q_1 = -1$, the statement is true for $n = 1$.

Let us assume that the statement is true for $n = m$, where m is a positive integer.

Then $p_mq_{m-1} - p_{m-1}q_m = (-1)^m$

$$\begin{aligned} p_{m+1}q_m - p_mq_{m+1} &= (a_{m+1}p_m + p_{m-1})q_m - p_m(a_{m+1}q_m - q_{m-1}) \\ &= p_{m-1}q_m - p_mq_{m-1} \\ &= -(-1)^m = (-1)^{m+1}. \end{aligned}$$

This shows that the statement is true for $n = m + 1$ if it is true for $n = m$. By the principle of induction the statement is true for all $n \in \mathbb{N}$.

7.3.3. If $\frac{p_r}{q_r}$ be the rth convergent of the simple continued fraction

$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$, then

(i) p_n is prime to q_n;

(ii) p_n is prime to p_{n-1};

(iii) q_n is prime to q_{n-1}.

Proof. (i) Let d be a positive common divisor of p_n and q_n.

Then $d \mid p_n$ and $d \mid q_n \Rightarrow d \mid (p_nq_{n-1} - p_{n-1}q_n)$, i.e., $d \mid (-1)^n$.

Consequently, $d = 1$ and therefore p_n is prime to q_n.

Similar proof for (ii) and (iii).

Note. Every convergent $\frac{p_r}{q_r}$ is in its lowest terms.

7.3.4. If $\frac{p_n}{q_n}$ be the nth convergent of the simple continued fraction

$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$ $a_1 > 0$,

then (i) $\frac{p_n}{p_{n-1}} = a_n + \frac{1}{a_{n-1}+}\cdots\frac{1}{a_2+}\frac{1}{a_1}$,

(ii) $\frac{q_n}{q_{n-1}} = a_n + \frac{1}{a_{n-1}+}\cdots\frac{1}{a_3+}\frac{1}{a_2}$.

Proof. $p_n = a_n p_{n-1} + p_{n-2}$ for $n > 2$

$$\frac{p_n}{p_{n-1}} = a_n + \frac{p_{n-2}}{p_{n-1}} = a_n + \frac{1}{\frac{p_{n-1}}{p_{n-2}}}$$

$$\frac{p_{n-1}}{p_{n-2}} = a_{n-1} + \frac{p_{n-3}}{p_{n-2}} = a_{n-1} + \frac{1}{\frac{p_{n-2}}{p_{n-3}}}$$

$$\cdots \qquad \cdots$$

$$\frac{p_3}{p_2} = a_3 + \frac{p_1}{p_2} = a_3 + \frac{1}{\frac{p_2}{p_1}}$$

$$\frac{p_2}{p_1} = \frac{a_2 a_1 + 1}{a_1} = a_2 + \frac{1}{a_1}.$$

Therefore $\frac{p_n}{p_{n-1}} = a_n + \frac{1}{a_{n-1} +} \cdots \frac{1}{a_2 +} \frac{1}{a_1}$.

$q_n = a_n q_{n-1} + q_{n-2}$ for $n > 2$.

$$\frac{q_n}{q_{n-1}} = \frac{a_n q_{n-1} + q_{n-2}}{q_{n-1}} = a_n + \frac{1}{\frac{q_{n-1}}{q_{n-2}}}$$

$$\frac{q_{n-1}}{q_{n-2}} = a_{n-1} + \frac{q_{n-3}}{q_{n-2}} = a_{n-1} + \frac{1}{\frac{q_{n-2}}{q_{n-3}}}$$

$$\cdots \qquad \cdots$$

$$\frac{q_3}{q_2} = a_3 + \frac{q_1}{q_2} = a_3 + \frac{1}{\frac{q_2}{q_1}}$$

$$\frac{q_2}{q_1} = a_2.$$

Therefore $\frac{q_n}{q_{n-1}} = a_n + \frac{1}{a_{n-1} +} \cdots \frac{1}{a_3 +} \frac{1}{a_2}$. □

7.3.5. (i) Let p, q, p', q' are positive integers and $pq' - p'q = 1$ and $q' < q$. If $\frac{p}{q}$ be expressed as a simple continued fraction with an even number of quotients then $\frac{p'}{q'}$ is the convergent immediately preceding $\frac{p}{q}$.

Proof. Let $\frac{p}{q} = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n}$.

Let $\frac{p''}{q''}$ be the convergent immediately preceding $\frac{p}{q}$. Since n is even, $pq'' - p''q = 1$. Therefore $pq' - p'q = pq'' - p''q$

or, $p(q' - q'') = q(p' - p'')$.

Since p, q are relatively prime, it follows that q divides $q' - q''$. Since $q' < q$, this cannot happen unless $q' = q''$. Consequently, $q' = q''$ and $p' = p''$. In other words $\frac{p'}{q'}$ is the convergent immediately preceding $\frac{p}{q}$.

(ii) Let p, q, p', q' are positive integers and $pq' - p'q = -1$ and $q' < q$. If $\frac{p}{q}$ be expressed as a simple continued fraction with an odd number of quotients then $\frac{p'}{q'}$ is the convergent immediately preceding $\frac{p}{q}$.

Similar proof.

Worked Examples.

1. If $\frac{p_n}{q_n}$ be the nth convergent of the simple continued fraction $a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots$ prove that

$p_n q_{n-3} - p_{n-3} q_n = (-1)^n [a_n a_{n-1} + 1]$, for $n \geq 3$.

$p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ $n \geq 2$

$$\begin{aligned}
&p_n q_{n-3} - p_{n-3} q_n \\
&= (a_n p_{n-1} + p_{n-2}) q_{n-3} - (a_n q_{n-1} + q_{n-2}) p_{n-3} \\
&= a_n [p_{n-1} q_{n-3} - q_{n-1} p_{n-3}] + (p_{n-2} q_{n-3} - p_{n-3} q_{n-2}) \\
&= a_n [(a_{n-1} p_{n-2} + p_{n-3}) q_{n-3} - (a_{n-1} q_{n-2} + q_{n-3}) p_{n-3}] + (-1)^{n-2} \\
&= a_n [a_{n-1} (p_{n-2} q_{n-3} - p_{n-3} q_{n-2})] + (-1)^{n-2} \\
&= a_n . a_{n-1} . (-1)^{n-2} + (-1)^n = (-1)^n [a_n a_{n-1} + 1].
\end{aligned}$$

2. If $\frac{p_r}{q_r}$ be the rth convergent of the simple continued fraction $a + \frac{1}{b +} \frac{1}{a +} \frac{1}{b +} \cdots$ prove that

(i) $p_{2n} = q_{2n+1}$, (ii) $bp_{2n-1} = aq_{2n}$.

(i) $\frac{p_{2n+2}}{q_{2n+2}} = a + \frac{1}{b + \frac{1}{\frac{p_{2n}}{q_{2n}}}} = a + \frac{p_{2n}}{bp_{2n} + q_{2n}} = \frac{(ab+1)p_{2n} + aq_{2n}}{bp_{2n} + q_{2n}}$

Therefore $p_{2n+2} = (ab + 1)p_{2n} + aq_{2n}$ ... (i)

and $q_{2n+2} = bp_{2n} + q_{2n}$ ... (ii)

$$\begin{aligned}
\text{From (i) } aq_{2n} &= p_{2n+2} - (ab+1)p_{2n} \\
&= (bp_{2n+1} + p_{2n}) - (ab+1)p_{2n} \\
&= bp_{2n+1} - abp_{2n} \\
&= b(ap_{2n} + p_{2n-1}) - abp_{2n} \\
&= bp_{2n-1}.
\end{aligned}$$

From (ii) $q_{2n} = q_{2n+2} - bp_{2n} = (bq_{2n+1} + q_{2n}) - bp_{2n}$

or, $p_{2n} = q_{2n+1}$.

3. If $\frac{p_n}{q_n}$ be the nth convergent of the simple continued fraction $3 + \frac{1}{6 +} \frac{1}{6 +} \cdots$ prove that

(i) $p_{n+1} + p_{n-1} = 20q_n$ (ii) $q_{n+1} + q_{n-1} = 2p_n$

(iii) $p_n^2 - 10q_n^2 = (-1)^n$.

Here $p_n = 6p_{n-1} + p_{n-2}$, $q_n = 6q_{n-1} + q_{n-2}$ for $n > 2$.

$$\frac{p_{n+1}}{q_{n+1}} = 3 + \frac{1}{3 + \frac{p_n}{q_n}} = 3 + \frac{q_n}{3q_n + p_n} = \frac{10q_n + 3p_n}{3q_n + p_n}.$$

Hence $p_{n+1} = 10q_n + 3p_n$ ... (A), $q_{n+1} = 3q_n + p_n$ ... (B)

(i) $20q_n = 2p_{n+1} - 6p_n$[using (A)]
$= p_{n+1} + (p_{n+1} - 6p_n)$
$= p_{n+1} + p_{n-1}$, since $p_{n+1} = 6p_n + p_{n-1}$.

(ii) $2p_n = 2q_{n+1} - 6q_n$[using (B)]
$= q_{n+1} + (q_{n+1} - 6q_n)$
$= q_{n+1} + q_{n-1}$, since $q_{n+1} = 6q_n + q_{n-1}$.

(iii) $10q_n = p_{n+1} - 3p_n$, using (A)
$p_n = q_{n+1} - 3q_n$, using (B).

Therefore $p_n^2 - 10q_n^2 = p_n[q_{n+1} - 3q_n] - q_n[p_{n+1} - 3p_n]$
$= -[p_{n+1}q_n - p_nq_{n+1}] = -(-1)^{n+1} = (-1)^n$.

4. If $\frac{p_n}{q_n}$ be the nth convergent of the continued fraction $\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\cdots$ prove that

(i) $p_n - (ab+2)p_{n-2} + p_{n-4} = 0$ for $n \geq 5$

(ii) $q_n - (ab+2)q_{n-2} + q_{n-4} = 0$ for $n \geq 5$

(iii) $ap_{2n+1} = bq_{2n}$ for $n \geq 1$.

(i) Here $p_1 = 0, q_1 = 1$; $p_2 = 1, q_2 = a$.

If n be odd, $p_n = bp_{n-1} + p_{n-2}$ for $n \geq 3$
$p_{n-1} = ap_{n-2} + p_{n-3}$ for $n \geq 5$
$p_{n-2} = bp_{n-3} + p_{n-4}$ for $n \geq 5$.

Therefore $p_n = b(ap_{n-2}+p_{n-3})+p_{n-2} = abp_{n-2}+(p_{n-2}-p_{n-4})+p_{n-2}$
$= (ab+2)p_{n-2} - p_{n-4}$.
or, $p_n - (ab+2)p_{n-2} + p_{n-4} = 0$ for $n \geq 5$ ... (A)

If n be even, $p_n = ap_{n-1} + p_{n-2}$ for $n \geq 4$
$p_{n-1} = bp_{n-2} + p_{n-3}$ for $n \geq 4$
$p_{n-2} = ap_{n-3} + p_{n-4}$ for $n \geq 6$.

Therefore $p_n = a(bp_{n-2}+p_{n-3})+p_{n-2} = abp_{n-2}+(p_{n-2}-p_{n-4})+p_{n-2}$
$= (ab+2)p_{n-2} - p_{n-4}$
or, $p_n - (ab+2)p_{n-2} + p_{n-4} = 0$ for $n \geq 6$ ... (B)

From (A) and (B) $p_n - (ab+2)p_{n-2} + p_{n-4} = 0$ for all $n \geq 5$.

(ii) Similarly, $q_n - (ab+2)q_{n-2} + q_{n-4} = 0$ for $n \geq 5$.

(iii) $\frac{p_{2n+2}}{q_{2n+2}} = \frac{bq_{2n}+p_{2n}}{(ab+1)q_{2n}+ap_{2n}}$.

Since each convergent is in its lowest terms, we have $p_{2n+2} = bq_{2n} + p_{2n}$. But $p_{2n+2} = ap_{2n+1} + p_{2n}$.
Therefore $bq_{2n} = ap_{2n+1}$.

7.4. Integral solution of the equation $ax + by = c$.

In Theorem 3.2.15 we have seen that the equation $ax + by = c$ where a, b, c are integers and $(a, b) \neq (0, 0)$, has an integral solution if and only if $gcd(a, b)$ is a divisor of c.

1. The equation $ax - by = 1$ where a and b are positive integers prime to each other.

Let $\frac{a}{b}$ be expressed as a simple continued fraction with an even number of quotients. Let $\frac{a'}{b'}$ be the convergent immediately preceding $\frac{a}{b}$. Then $ab' - a'b = 1$.

This shows that (b', a') is a solution of the equation.

Also we have $ax - by = ab' - a'b$
or, $a(x - b') = b(y - a')$.

Since a and b are prime to each other, b is a divisor of $x - b'$ and a is a divisor of $y - a'$.

Therefore $\frac{x-b'}{b} = \frac{y-a'}{a} = t$, where t is an integer
or, $x = bt + b'$
$y = at + a'$, where $t = 0, \pm1, \pm2, \cdots$
This gives the general solution in integers.

Note. If c be a positive integer, then the general solution of the equation $ax - by = c$ where a, b are positive integers prime to each other is given by $x = bt + b'c$
$y = at + a'c$, where $t = 0, \pm1, \pm2, \cdots$

2. The equation $ax - by = -1$ where a and b are positive integers prime to each other.

Let $\frac{a}{b}$ be expressed as a simple continued fraction with an odd number of quotients. Let $\frac{a'}{b'}$ be the convergent immediately preceding $\frac{a}{b}$. Then $ab' - a'b = -1$.

This shows that (b', a') is a solution of the equation.

Also we have $ax - by = ab' - a'b$
or, $a(x - b') = b(y - a')$.

Since a and b are prime to each other, b is a divisor of $x - b'$ and a is a divisor of $y - a'$.

Therefore $\frac{x-b'}{b} = \frac{y-a'}{a} = t$, where t is an integer.
or, $x = bt + b'$
$y = at + a'$, where $t = 0, \pm1, \pm2, \cdots$
This gives the general solution in integers.

3. The equation $ax + by = 1$ where a and b are positive integers prime to each other.

Let $\frac{a}{b}$ be expressed as a simple continued fraction with an even number of quotients. Let $\frac{a'}{b'}$ be the convergent immediately preceding $\frac{a}{b}$. Then $ab' - a'b = 1$.

This shows that $(b', -a')$ is a solution of the equation.

Also we have $ax + by = ab' - a'b$

or, $a(x - b') = b(-y - a')$.

Since a and b are prime to each other, b is a divisor of $x - b'$ and a is a divisor of $-y - a'$.

Therefore $\frac{x-b'}{b} = \frac{y+a'}{-a} = t$, where t is an integer.

or, $x = bt + b'$

$y = -at - a'$, where $t = 0, \pm 1, \pm 2, \cdots$

This gives the general solution in integers.

Note. If c be a positive integer, then the general solution of the equation $ax + by = c$ where a, b are positive integers prime to each other is given by $x = bt + b'c$

$y = -at - a'c$, where $t = 0, \pm 1, \pm 2, \cdots$

4. The equation $ax + by = -1$ where a and b are positive integers prime to each other.

Let $\frac{a}{b}$ be expressed as a simple continued fraction with an odd number of quotients. Let $\frac{a'}{b'}$ be the convergent immediately preceding $\frac{a}{b}$. Then $ab' - a'b = -1$.

This shows that $(b', -a')$ is a solution of the equation.

Proceeding as in 3, the general solution in integers is given by

$x = bt + b'$

$y = -at - a'$, where $t = 0, \pm 1, \pm 2, \cdots$

Worked Examples.

1. Solve the equation $14x - 19y = 1$ in integers.

Let us express $\frac{14}{19}$ as a simple continued fraction with an even number of quotients.

$$\frac{14}{19} = 0 + \frac{1}{1+} \frac{1}{2+} \frac{1}{1+} \frac{1}{3+} \frac{1}{1}.$$

Here the convergents are $\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{3}{4}, \frac{11}{15}, \frac{14}{19}$

The convergent immediately preceding $\frac{14}{19}$ is $\frac{11}{15}$

We have $14.15 - 19.11 = 1$ .... (A)

Hence $14x - 19y = 14.15 - 19.11$

or, $14(x - 15) = 19(y - 11)$

Since 14 and 19 are prime to each other, 19 is a divisor of $x - 15$ and 14 is a divisor of $y - 11$.

Therefore $\frac{x-15}{19} = \frac{y-11}{14} = t$, where t is an integer.

The general solution is $x = 19t + 15$

$y = 14t + 11$, where $t = 0, \pm 1, \pm 2, \cdots$

Note. From (A) it follows that $(15, 11)$ is a solution of the equation. This solution is given by the convergent immediately preceding $\frac{14}{19}$.

2. Solve the equation $14x - 19y = -1$ in integers.

Let us express $\frac{14}{19}$ as a continued fraction with an odd number of quotients.

$$\frac{14}{19} = 0 + \frac{1}{1+} \frac{1}{2+} \frac{1}{1+} \frac{1}{4}$$

Here the convergents are $\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{3}{4}, \frac{14}{19}$.

The convergent immediately preceding $\frac{14}{19}$ is $\frac{3}{4}$.

We have $14.4 - 19.3 = -1$ .... (A)

Hence $14x - 19y = 14.4 - 19.3$

or, $14(x - 4) = 19(y - 3)$

Since 14 and 19 are prime to each other, 19 is a divisor of $x - 4$ and 14 is a divisor of $y - 3$.

Therefore $\frac{x-4}{19} = \frac{y-3}{14} = t$, where t is an integer.

The general solution is $x = 19t + 4$

$y = 14t + 3$ where $t = 0, \pm 1, \pm 2, \cdots$

Note. From (A) it follows that $(4, 3)$ is a solution of the equation. This solution is given by the convergent immediately preceding $\frac{14}{19}$.

3. Solve the equation $14x - 19y = 5$ in integers.

Proceeding as in Example 1, we have $14.15 - 19.11 = 1$.

Hence $14x - 19y = 5(14.15 - 19.11)$

or, $14(x - 75) = 19(y - 55)$

Since 14 and 19 are prime to each other, 19 is a divisor of $x - 75$ and 14 is a divisor of $y - 55$.

Therefore $\frac{x-75}{19} = \frac{y-55}{14} = t$, where t is an integer.

The general solution is $x = 19t + 75$

$y = 14t + 55$ where $t = 0, \pm 1, \pm 2, \cdots$

Note. The least solution in positive integers corresponds to $t = -3$. The least solution is (18, 13).

7.5. Properties of a simple continued fraction.

Theorem 7.5.1. An infinite simple continued fraction is convergent.

Proof. Let $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$ be an infinite simple continued fraction.

Let $u_n(=\frac{p_n}{q_n})$ be the nth convergent of the continued fraction.

Now $u_n - u_{n-1} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_n q_{n-1}}$

$$u_n - u_{n-2} = \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^{n-1}a_n}{q_n q_{n-2}}.$$

Since $q_n, q_{n-1}, q_{n-2}, a_n$ are all positive, it follows that $u_n - u_{n-1}$ and $u_n - u_{n-2}$ are of opposite signs.

Therefore u_n lies between two preceding convergents u_{n-1} and u_{n-2}.

But $u_1 = a_1, \quad u_2 = a_1 + \frac{1}{a_2} > u_1$.

$u_1 < u_2 \Rightarrow u_1 < u_3 < u_2$

$u_3 < u_2 \Rightarrow u_3 < u_4 < u_2$

$u_3 < u_4 \Rightarrow u_3 < u_5 < u_4$

$u_5 < u_4 \Rightarrow u_5 < u_6 < u_4$

$\cdots \quad \cdots$

Hence $u_1 < u_3 < u_5 < \cdots < u_4 < u_6 < u_2$.

Thus the sequence of odd convergents $\{u_{2n-1}\}$ is a strictly increasing sequence, bounded above and the sequence of even convergents $\{u_{2n}\}$ is a strictly decreasing sequence, bounded below.

Therefore both the sequences are convergent.

But $u_{2n} - u_{2n-1} = \frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}} = \frac{1}{q_{2n}q_{2n-1}}$.

As $q_n \to \infty$ as $n \to \infty$, $\lim(u_{2n} - u_{2n-1}) = 0$. So $\lim u_{2n} = \lim u_{2n-1} = l$, say. Therefore both the sequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converge to the same limit l and the sequence $\{u_n\}$ converges to l.

Since l is the limit of the strictly increasing sequence $\{u_{2n-1}\}$, l is the lub of the sequence $\{u_{2n-1}\}$.

Since l is the limit of the strictly decreasing sequence $\{u_{2n}\}$, l is the glb of the sequence $\{u_{2n}\}$.

Therefore $u_1 < u_3 < u_5 < \cdots < l < \cdots < u_6 < u_4 < u_2$.

Hence every infinite simple continued fraction is convergent and

(i) its value is greater than every odd convergent,

(ii) its value is less than every even convergent.

This completes the proof.

Let us consider the simple continued fraction

$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$

If this be a finite continued fraction then it has a value which is equal to its last convergent.

If, however, this be an infinite continued fraction then also it has a value, by the previous theorem.

Let $F = a_1 + \frac{1}{a_2+}\frac{1}{a_3}\cdots$

$\alpha_2 = a_2 + \frac{1}{a_3+}\frac{1}{a_4+}\cdots$

$\alpha_3 = a_3 + \frac{1}{a_4+}\frac{1}{a_5+}\cdots$

$\cdots \quad \cdots$

α_i is called the ith *complete quotient* $(i \geq 2)$ of the continued fraction.

Therefore we have $F = a_1 + \frac{1}{\alpha_2}, F = a_1 + \frac{1}{a_2+}\frac{1}{\alpha_3}, F = a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{\alpha_4}, \ldots$ That is, the value of the continued fraction is obtained from its nth convergent by substituting α_n for a_n.

Since $\frac{p_n}{q_n} = \frac{a_n p_{n-1}+p_{n-2}}{a_n q_{n-1}+q_{n-2}}$, it follows that $F = \frac{\alpha_n p_{n-1}+p_{n-2}}{\alpha_n q_{n-1}+q_{n-2}}$.

Theorem 7.5.2. Each convergent is a closer approximation to the value of a simple continued fraction than the preceding.

Proof. Let F be the value of the simple continued fraction

$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$

Let $\frac{p_n}{q_n}$ be the nth convergent. Then $\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}p_n+p_{n-1}}{a_{n+1}q_n+q_{n-1}}$.

Let λ be the $(n+1)$th complete quotient of the continued fraction.

Then $\lambda = a_{n+1} + \frac{1}{a_{n+2}+}\frac{1}{a_{n+3}+}\cdots$.

The value of the continued fraction can be obtained from $\frac{p_{n+1}}{q_{n+1}}$ if a_{n+1} be replaced by λ. Therefore $F = \frac{\lambda p_n + p_{n-1}}{\lambda q_n + q_{n-1}}$.

$$F - \frac{p_n}{q_n} = \frac{\lambda p_n+p_{n-1}}{\lambda q_n+q_{n-1}} - \frac{p_n}{q_n} = \frac{-(p_n q_{n-1}-p_{n-1}q_n)}{(\lambda q_n+q_{n-1})q_n} = \frac{(-1)^{n+1}}{(\lambda q_n+q_{n-1})q_n}.$$

$$F - \frac{p_{n-1}}{q_{n-1}} = \frac{\lambda p_n+p_{n-1}}{\lambda q_n+q_{n-1}} - \frac{p_{n-1}}{q_{n-1}}$$

$$= \frac{\lambda(p_n q_{n-1}-p_{n-1}q_n)}{(\lambda q_n+q_{n-1})q_{n-1}} = \frac{(-1)^n\lambda}{(\lambda q_n+q_{n-1})q_{n-1}}.$$

Therefore $\frac{|z-\frac{p_n}{q_n}|}{|z-\frac{p_{n-1}}{q_{n-1}}|} = \frac{q_{n-1}}{\lambda q_n} < 1$, since $q_n > q_{n-1}, \lambda > 1$

or, $|F - \frac{p_n}{q_n}| < |z - \frac{p_{n-1}}{q_{n-1}}|$.

It follows that F is nearer to $\frac{p_n}{q_n}$ than to $\frac{p_{n-1}}{q_{n-1}}$.

Theorem 7.5.3. Any convergent of a simple continued fraction
$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$
is a closer approximation to the value of the continued fraction than any rational number $\frac{r}{s}$ whose denominator $s < q_n$.

Proof. Let F be the value of the continued fraction.

Then F lies between $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$ and $|F - \frac{P}{n}| < |F - \frac{p_{n-1}}{q_{n-1}}|$

If F be nearer to $\frac{r}{s}$ than to $\frac{p_n}{q_n}$, then
$|F - \frac{r}{s}| < |F - \frac{p_n}{q_n}| < |F - \frac{p_{n-1}}{q_{n-1}}|$.

Since F lies between $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$, it follows that $\frac{r}{s}$ lies between $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$.

Hence $|\frac{r}{s} - \frac{p_{n-1}}{q_{n-1}}| < |\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}|$

or, $\frac{|rq_{n-1} - sp_{n-1}|}{sq_{n-1}} < \frac{1}{q_n q_{n-1}}$

or, $s > q_n\{|rq_{n-1} - sp_{n-1}|\}$.

But $|rq_{n-1} - sp_{n-1}|$ is a positive integer and therefore $s > q_n$, a contradiction to the condition on s.

Hence $|F - \frac{r}{s}| > |F - \frac{p_n}{q_n}|$ and the theorem is done.

Theorem 7.5.4. If ϵ be the error (numerical value) in taking $\frac{p_n}{q_n}$ for the value of the simple continued fraction
$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$ then
$$\frac{1}{2q_{n+1}^2} < \frac{1}{q_n(q_{n+1}+q_n)} < \epsilon < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.$$

Proof. Let F be the value of the continued fraction and λ be the $(n+2)$th complete quotient of the continued fraction.

Then $F = \frac{\lambda p_{n+1}+p_n}{\lambda q_{n+1}+q_n}$, $\lambda > 1$.

$\epsilon = |F - \frac{p_n}{q_n}| = |\frac{\lambda p_{n+1}+p_n}{\lambda q_{n+1}+q_n} - \frac{p_n}{q_n}| = \frac{\lambda}{q_n(\lambda q_{n+1}+q_n)} = \frac{1}{q_n(q_{n+1}+\frac{q_n}{\lambda})}$.

But $\lambda > 1 \Rightarrow 0 < \frac{q_n}{\lambda} < q_n$.

Therefore $0 < q_{n+1} < q_{n+1} + \frac{q_n}{\lambda} < q_{n+1} + q_n$ and

$$\frac{1}{q_n(q_{n+1}+q_n)} < \frac{1}{q_n(q_{n+1}+\frac{q_n}{\lambda})} < \frac{1}{q_n q_{n+1}}$$

i.e., $\frac{1}{q_n(q_{n+1}+q_n)} < \epsilon < \frac{1}{q_n q_{n+1}}$.

Again $q_{n+1} > q_n \Rightarrow q_n q_{n+1} > q_n^2$ and $q_n(q_{n+1} + q_n) < q_{n+1}(q_{n+1} + q_{n+1}) = 2q_{n+1}^2$.

Therefore $\frac{1}{2q_{n+1}^2} < \frac{1}{q_n(q_{n+1}+q_n)} < \epsilon < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$.

Note 1. $q_{n+1} = a_{n+1}q_n + q_{n-1} > a_{n+1}q_n$.
Therefore $q_n q_{n+1} > a_{n+1}q_n^2$. Hence $\epsilon < \frac{1}{q_n q_{n+1}} < \frac{1}{a_{n+1}q_n^2}$.

It follows that if a_{n+1} be a comparatively large quotient, $\frac{p_n}{q_n}$ is a very good approximation to the value of the continued fraction, i.e., any convergent which immediately preceds a comparatively large quotient is a very good approxiation.

Note 2. Since $\epsilon < \frac{1}{q_n^2}$, it follows that in order to find a convergent which differs from the value of the continued fraction by less than a given quantity $\frac{1}{\alpha}$, we have only to calculate convergents upto $\frac{p_n}{q_n}$ where $q_n > \sqrt{\alpha}$.

Worked Examples.

1. Given that 1 kilogram = 2.2046 pounds show by the theory of continued fraction that 44 kgs is slightly greater than 97 pounds.

1 kg = $\frac{22046}{10000}$ pounds.

$\frac{22046}{10000} = 2+\frac{1}{\frac{10000}{2046}}$, $\frac{10000}{2046} = 4+\frac{1}{\frac{2046}{1816}}$, $\frac{2046}{1816} = 1+\frac{1}{\frac{1816}{230}}$,

$\frac{1816}{230} = 7+\frac{1}{\frac{230}{206}}$, $\frac{230}{206} = 1+\frac{1}{\frac{206}{24}}$, ...

Therefore $\frac{22046}{10000} = 2 + \frac{1}{4+}\frac{1}{1+}\frac{1}{7+}\frac{1}{1+}\cdots$

The first convergent = 2, the second convergent = $\frac{9}{4}$,

the third convergent = $\frac{1.9+2}{1.4+1} = \frac{11}{5}$,

the fourth convergent = $\frac{7.11+9}{7.5+4} = \frac{86}{39}$,

the fifth convergent = $\frac{1.86+11}{1.39+5} = \frac{97}{44}$.

The fifth convergent is approximately equal to $\frac{22046}{10000}$ and since it is an odd convergent, $\frac{97}{44} < \frac{22046}{10000}$.

Hence 1 kg is approximately equal to $\frac{97}{44}$ pounds and 1 kg > $\frac{97}{44}$ pounds. It follows that 44 kg is approximately equal to 97 pounds and 44 kg > 97 pounds.

2. Find an approximation to the value of the continued fraction $1 + \frac{1}{3+}\frac{1}{5+}\frac{1}{7+}\cdots$ which differ from the value by less than .0001.

Here $\frac{p_1}{q_1} = 1$, $\frac{p_2}{q_2} = \frac{4}{3}$, $\frac{p_3}{q_3} = \frac{5.4+1}{5.3+1} = \frac{21}{16}$, $\frac{p_4}{q_4} = \frac{7.21+4}{7.16+3} = \frac{151}{115}$, $\frac{p_5}{q_5} = \frac{9.151+21}{9.115+16} = \frac{1380}{1051}$, ...

Let ϵ be the error in taking $\frac{p_n}{q_n}$ as the value of the continued fraction. Then $\epsilon < \frac{1}{q_n^2}$. If $q_n \geq 100$ then $\epsilon < \frac{1}{10^4}$.

Here $q_4 > 100$. Therefore $\frac{p_4}{q_4}$ is the required approximation.

3. The sidereal period of venus and the earth are 224.7 days and 365.25 days respectively. Find various cycles in which transit of venus may occur.

Let the transit occurs after x sidereal periods of venus and y sidereal period of the earth, where x, y are positive integers.

Then $224.7x = 365.25y$

or, $\frac{x}{y} = \frac{365.25}{224.7} = 1+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{30+}\frac{1}{1+}\frac{1}{15}$.

Let $\frac{p_r}{q_r}$ be the rth convergent of the simple continued fraction.

Then $p_1 = 1, \quad p_2 = 2, \quad p_3 = 3, \quad p_4 = 5, \quad p_5 = 13, \quad p_6 = 395.$

$q_1 = 1, \quad q_2 = 1, \quad q_3 = 2, \quad q_4 = 3, \quad q_5 = 8, \quad q_6 = 243.$

The 5th convergent $\frac{p_5}{q_5}$ precedes a large quotient.

Therefore $\frac{p_5}{q_5}$ is a very good approximation. If ϵ be the error in taking $\frac{13}{8}$ as the value of the continued fraction,

$\epsilon < \frac{1}{8.243}$, i.e., $\epsilon < .0005$.

Therefore 13 sidereal preiods of venus $\simeq$ 8 sidereal periods of the earth. In other words, the transit of venus will occur after every 8 years.

Again, $\frac{p_6}{q_6}$ is a better approximation to the value of the continued fraction. Hence the transit will occur after every 243 years, more accurately.

Theorem 7.5.5. Let $\frac{p_r}{q_r}$ be the rth convergent of a simple continued fraction whose value is F then

$F^2 >$ or $< \frac{p_n}{q_n}.\frac{p_{n-1}}{q_{n-1}}$ according as n is even or odd.

Proof. Let $F = a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$

Let λ be the $(n+1)$th complete quotient. Then $F = \frac{\lambda p_n + p_{n-1}}{\lambda q_n + q_{n-1}}$.

Now $F^2 - \frac{p_n}{q_n}.\frac{p_{n-1}}{q_{n-1}} = \left(\frac{\lambda p_n + p_{n-1}}{\lambda q_n + q_{n-1}}\right)^2 - \frac{p_n p_{n-1}}{q_n q_{n-1}}$

$= \frac{(\lambda^2 p_n^2 + 2\lambda p_n p_{n-1} + p_{n-1}^2) q_n q_{n-1} - (\lambda^2 q_n^2 + 2\lambda q_n q_{n-1} + q_{n-1}^2) p_n p_{n-1}}{(\lambda q_n + q_{n-1})^2 q_n q_{n-1}}$

$= \frac{\lambda^2 p_n q_n . (-1)^n + p_{n-1} q_{n-1} (-1)^{n+1}}{(\lambda q_n + q_{n-1})^2 q_n q_{n-1}}$

$= (-1)^n . \frac{\lambda^2 p_n q_n - p_{n-1} q_{n-1}}{(\lambda q_n + q_{n-1})^2 q_n q_{n-1}}$.

Therefore $F^2 - \frac{p_n}{q_n}.\frac{p_{n-1}}{q_{n-1}} >$ or < 0 according as n is even or odd, since $\lambda^2 > 1, p_n > p_{n-1}, q_n > q_{n-1}$.

or, $F^2 >$ or $< \frac{p_n}{q_n}.\frac{p_{n-1}}{q_{n-1}}$ according as n is even or odd.

Note. If $\frac{p}{q}$ be an odd convergent and $\frac{X}{Y}$ be the even convergent immediately preceding $\frac{p}{q}$, then $F^2 < \frac{pX}{qY}$.

If $\frac{p}{q}$ be an even convergent and $\frac{X}{Y}$ be the odd convergent immediately preceding $\frac{p}{q}$, then $F^2 > \frac{pX}{qY}$.

7.6. Recurring simple continued fraction.

A simple continued fraction

$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$

is said to be a *recurring continued fraction* if after a certain stage the elements recur in the same order. The recurring elements form the *cyclic part* and the non-recurring elements, if they exist, form the *acyclic part* of the continued fraction.

For example, the continued fraction

$1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\frac{1}{3+}\frac{1}{4+}\cdots$

is a recurring continued fraction. Here $1 + \frac{1}{2}$ is the acyclic part and $\frac{1}{3+}\frac{1}{4}$ is the cycle.

The cycle is denoted by putting * under the first and the last element of the cycle.

The recurring continued fraction $1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\frac{1}{3+}\frac{1}{4+}\cdots$ is denoted by $1+\frac{1}{2+}\underset{*}{\frac{1}{3+}}\underset{*}{\frac{1}{4+}}$

Worked Examples.

1. If $F = \frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\cdots$ show that the roots of the equation $x^2 - (ab+2)x = 1 = 0$ are $1 + ab + aF$ and $1 - aF$.

$F = \frac{1}{a+}\frac{1}{b+F} = \frac{1}{a+\frac{1}{b+F}} = \frac{b+F}{ab+aF+1}$

or, $aF^2 + abF - b = 0$.

Therefore $F = \frac{-ab+\sqrt{a^2b^2+4ab}}{2a}$, since F is positive.

or, $2aF + ab = \sqrt{a^2b^2 + 4ab}$.

The roots of the equation $x^2-(ab+2)x+1=0$ are $\frac{ab+2+\sqrt{a^2b^2+4ab}}{2}$ and $\frac{ab+2-\sqrt{a^2b^2+4ab}}{2}$

i.e., $\frac{ab+2+2aF+ab}{2}$ and $\frac{ab+2-2aF-ab}{2}$

i.e., $1+ab+F$ and $1-aF$.

2. Prove that $\frac{1}{a}\frac{1}{b}\frac{1}{c}\sim\frac{1}{b}\frac{1}{a}\frac{1}{c}=\frac{a-b}{1+ab}$.

Let $x=\frac{1}{a}\frac{1}{b}\frac{1}{c}$.

Then $x=\frac{1}{a+}\frac{1}{b+}\frac{1}{c+x}=\frac{bc+bx+1}{abc+abx+a+c+x}$

or, $(ab+1)x^2+(abc+a-b+c)x-(bc+1)=0$ ... (i)

Let $y=\frac{1}{b}\frac{1}{a}\frac{1}{c}$.

Then y is obtained from x by replacing a by b and b by a.

Therefore $(ab+1)y^2+(abc-a+b+c)y-(ac+1)=0$ ... (ii)

From (i) and (ii)

$(ab+1)(x^2-y^2)+(abc+c)(x-y)+(a-b)(x+y)+(a-b)c=0$

or, $(ab+1)(x-y)[x+y+c]+(a-b)(x+y+c)=0$

or, $(ab+1)(x-y)+a-b=0$, since $x+y+c\neq 0$

or, $x\sim y=\frac{a-b}{ab+1}$.

3. If the continued fractions

$x+\frac{1}{x+}\frac{1}{x+}\cdots,\quad y+\frac{1}{y+}\frac{1}{y+}\cdots,\quad x+\frac{1}{x+}\frac{1}{x+}\cdots$

be in geometric progression of common ratio r, prove that $r+\frac{1}{r}=\frac{z+x}{y}$.

Let $F=x+\frac{1}{x+}\frac{1}{x+}\cdots,\quad P=y+\frac{1}{y+}\frac{1}{y+}\cdots,\quad Q=z+\frac{1}{z+}\frac{1}{z+}\cdots$

Then $F=x+\frac{1}{F},\quad P=y+\frac{1}{P},\quad Q=x+\frac{1}{Q}$

or, $x=F-\frac{1}{F},\quad y=P-\frac{1}{P},\quad z=Q-\frac{1}{Q}$.

By the given condition, $P=rF, Q=r^2F$.

$z+x=(Q+F)-\frac{1}{Q}-\frac{1}{F}=(r^2+1)F-\frac{1}{r^2F}-\frac{1}{F}=(r^2+1)F-\frac{r^2+1}{r^2}\cdot\frac{1}{F}$

$=(r^2+1)(F-\frac{1}{r^2F})$

$y=rF-\frac{1}{rF}=r(F-\frac{1}{r^2F})$.

Therefore $\frac{z+x}{y}=\frac{r^2+1}{r}=r+\frac{1}{r}$.

4. If $x=a+\frac{1}{b+}\frac{1}{c+},\quad y=b+\frac{1}{c+}\frac{1}{a+},\quad z=c+\frac{1}{a+}\frac{1}{b+}$

prove that $xyz=t+\frac{1}{t+}\frac{1}{t+}\cdots$, where $t=abc+a+b+c$.

$x=a+\frac{1}{y},\quad y=b+\frac{1}{z},\quad z=c+\frac{1}{x}$.

Therefore $xyz=abc+\frac{a}{zx}+\frac{b}{xy}+\frac{c}{yz}+\frac{ab}{z}+\frac{bc}{y}+\frac{ca}{z}+\frac{1}{xyz}$

$x=a+\frac{1}{y}\Rightarrow b=\frac{ab}{x}+\frac{b}{xy}$

$y=b+\frac{1}{z}\Rightarrow c=\frac{bc}{y}+\frac{c}{yz}$

$z=c+\frac{1}{x}\Rightarrow a=\frac{ca}{z}+\frac{a}{zx}$

Therefore $xyz = abc+a+b+c+\frac{1}{xyz}$

$= t+\frac{1}{xyz}$

$= t+\frac{1}{t+}\frac{1}{t+}\cdots$, where $t=abc+a+b+c$.

5. Show that the nth convergent of the continued fraction

$2+\frac{1}{2+}\frac{1}{2+}\cdots$ is $\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^n-(1-\sqrt{2})^n}$.

Let $\frac{p_n}{q_n}$ be the nth convergent. Then $p_n=2p_{n-1}+p_{n-2}, q_n=2q_{n-1}+q_{n-2}, n>2$ and $p_1=2, q_1=1, p_2=5, q_2=2$.

Since $p_n-2p_{n-1}-p_{n-2}=0, p_n=A\alpha^n+B\beta^n$ where α,β are the roots of the equation $x^2-2x-1=0$.

The roots of the equation are $1\pm\sqrt{2}$. Let $\alpha=1+\sqrt{2}, \beta=1-\sqrt{2}$.

Then $\alpha^2=2\alpha+1, \beta^2=2\beta+1, \alpha+\beta=2, \alpha\beta=-1, \alpha-\beta=2\sqrt{2}$.

Since $p_1=2, p_2=5, A\alpha+B\beta-2=0, A\alpha^2+B\beta^2-5=0$.

Therefore $A=\frac{2\beta^2-5\beta}{\alpha\beta(\beta-\alpha)}=\frac{(4\beta+2)-5\beta}{-(\beta-\alpha)}=\frac{2-\beta}{\alpha-\beta}=\frac{\sqrt{2}+1}{2\sqrt{2}}$,

and $B=\frac{5\alpha-2\alpha^2}{\alpha\beta(\beta-\alpha)}=\frac{5\alpha-(4\alpha+2)}{-(\beta-\alpha)}=\frac{\alpha-2}{\alpha-\beta}=-\frac{1-\sqrt{2}}{2\sqrt{2}}$.

Hence $p_n=\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}$.

Since $q_n-2q_{n-1}-q_{n-2}=0, q_n=A\alpha^n+B\beta^n$ where α,β are the roots of the equation $x^2-2x-1=0$.

Since $q_1=1, q_2=2, A\alpha+B\beta-1=0, A\alpha^2+B\beta^2-2=0$.

Therefore $A=\frac{2\beta^2-2\beta}{\alpha\beta(\beta-\alpha)}\frac{1}{\alpha-\beta}=\frac{1}{2\sqrt{2}}, \quad B=\frac{2\alpha-\alpha^2}{\alpha\beta(\beta-\alpha)}\frac{1}{\beta-\alpha}=-\frac{1}{2\sqrt{2}}$.

Hence $q_n=\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$.

Consequently, the nth convergent is $\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^n-(1-\sqrt{2})^n}$.

7.7. Symmetric continued fraction.

A finite simple continued fraction is said to be *symmetric* if the quotients equidistant from the beginning and the end are equal.

For example, the continued fractions
$1+\frac{1}{2+}\frac{1}{2+}\frac{1}{1}$, $\quad 2+\frac{1}{3+}\frac{1}{4+}\frac{1}{3+}\frac{1}{2}$ are symmetric continued fractions.

Worked Examples.

1. If $\frac{P}{Q}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_r+}\frac{1}{a_r+}\cdots\frac{1}{a_1}$ and $\frac{P}{Q}$ be in its lowest terms prove that Q^2+1 is divisible by P.

$\frac{P}{Q}$ is a symmetric continued fraction having an even number of quotients. Let $\frac{P'}{Q'}$ be the convergent immediately preceding $\frac{P}{Q}$.

By 7.3.4, $\frac{P}{P'}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_r+}\frac{1}{a_r+}\cdots\frac{1}{a_1}=\frac{P}{Q}$. Therefore $P'=Q$.

$PQ'-P'Q=(-1)^{2r}=1$ or, $PQ'=1+P'Q=1+Q^2$.

Therefore $1+Q^2$ is divisible by P.

2. If $\frac{P}{Q}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_{r-1}+1}\frac{1}{a_r+}\frac{1}{a_{r-1}+}\cdots\frac{1}{a_1}$ and $\frac{P}{Q}$ be in its lowest terms prove that Q^2-1 is divisible by P.

$\frac{P}{Q}$ is a symmetric continued fraction having an odd number of quotients. Let $\frac{P'}{Q'}$ be the convergent immediately preceding $\frac{P}{Q}$.

By 7.3.4, $\frac{P}{P'}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_{r-1}+}\frac{1}{a_r+}\cdots\frac{1}{a_1}=\frac{P}{Q}$. Therefore $P'=Q$.

$PQ'-P'Q=(-1)^{2r+1}=-1$ or, $PQ'=P'Q-1=Q^2-1$.

Therefore Q^2-1 is divisible by P.

Exercises 7

1. Express r as a finite simple continued fraction having an odd number of quotients where $r=$

(i) $\frac{41}{15}$, (ii) $\frac{71}{25}$, (iii) $\frac{61}{23}$, (iv) $\frac{14}{19}$.

2. Express r as a finite simple continued fraction having an even number of quotients where $r=$

(i) $\frac{41}{15}$, (ii) $\frac{71}{25}$, (iii) $\frac{61}{23}$, (iv) $\frac{14}{19}$.

3. Find two positive integers m, n such that

(i) $11m-15n=1$, (ii) $11m-15n=-1$, (iii) $25m-39n=1$.

4. Solve the equation in integers

(i) $41x-15y=1$, (ii) $41x-15y=-1$, (iii) $41x+15y=1$,
(iv) $41x+15y=-1$, (v) $14x-19y=1$, (vi) $14x-19y=-1$,
(vii) $14x+19y=1$, (viii) $14x+19y=-1$.

5. Express α as a simple continued fraction where $\alpha=$

(i) $\sqrt{2}$, (ii) $\sqrt{7}$, (iii) $\sqrt{11}$,
(iv) $\sqrt{14}$, (v) $\frac{\sqrt{37}+8}{9}$, (vi) $\frac{\sqrt{37}+4}{7}$.

6. A line segment AB is divided internally at C so that $AB.AC=BC^2$. Express $\frac{AC}{AB}$ as a simple continued fraction.

7. Prove that

(i) $3+\frac{1}{2+}\frac{1}{6+}\frac{1}{2+}\frac{1}{6+}\cdots=2(1+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\cdots)$

(ii) $4+\frac{1}{4+}\frac{1}{8+}\frac{1}{4+}\frac{1}{8+}\cdots=3(1+\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\cdots)$

(ii) $3+\frac{1}{1+}\frac{1}{6+}\frac{1}{1+}\frac{1}{6+}\cdots=3(1+\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\cdots)$

8. Given that 1 metre = 3.2809 ft show by the theory of continued fraction that

(i) 8 kms $\simeq$ 5 miles and 8 kms $<$ 5 miles;

(ii) 103 kms $\simeq$ 64 miles and 103 kms$>$ 64 miles.

9. (i) Express $\sqrt{10}$ as a simple continued fraction. Show that $\frac{199}{60}$ is greater than $\sqrt{10}$ and it differs from $\sqrt{10}$ by less than $\frac{1}{20000}$.

(ii) Express $\sqrt{14}$ as a simple continued fraction. Show that $\frac{449}{120}$ is greater than $\sqrt{14}$ and it differs from $\sqrt{14}$ by less than $\frac{1}{90000}$.

10. (i) Given that π is approximately equal to 3.14159, show by the theory of continued fraction that $\frac{355}{113}$ is an approximation to π with an error less than 4×10^{-6}. Show also that the error is in excess.

(ii) Given that $\pi=3+\frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292+}\frac{1}{1+}\cdots$, show that $\frac{355}{113}$ is an approximation to π with an error less than 3×10^{-7}. Show also that the error is in excess.

11. (i) Show by the theory of continued fraction that $\frac{171}{77}$ differs from 2.2208 by a quantity less than $\frac{1}{48125}$. Show also that $\frac{171}{77}$ is less than 2.2208.

(ii) Show by the theory of continued fraction that $\frac{173}{69}$ differs from 2.5072 by a quantity less than $\frac{1}{19182}$. Show also that $\frac{173}{69}$ is greater than 2.5072.

12. Let $P>Q>0$ and P be prime to Q. Let $\frac{P}{Q}$ be converted into a simple continued fraction and let a be the first convergent and $\frac{p}{q}$ be the convergent immediately preceding $\frac{P}{Q}$. Prove that if $\frac{Q}{q}$ be converted into a simple continued fraction the convergent immediately preceding $\frac{Q}{q}$ is $\frac{P-aQ}{p-aq}$.

13. Let $p > q$ and p be prime to q. If $\frac{p}{q} = a_1 + \frac{1}{a_2+}\cdots\frac{1}{a_{n-1}+}\frac{1}{a_n}$, show that $\frac{1}{a_n+}\frac{1}{a_{n-1}+}\cdots\frac{1}{a_1} \sim \frac{1}{a_n+}\frac{1}{a_{n-1}+}\cdots\frac{1}{a_2} = \frac{1}{pq}$.

14. Show that the difference the between the first and the nth convergent of the simple continued fraction $a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots$ is numerically equal to $\frac{1}{q_1q_2} - \frac{1}{q_2q_3} + \cdots + (-1)^n\frac{1}{q_{n-1}q_n}$.

15. If a be a positive integer, show that

(i) $\sqrt{a^2+1} = a + \frac{1}{2a+}\frac{1}{2a+}\frac{1}{2a+}\cdots$,

(ii) $\sqrt{a^2+2} = a + \frac{1}{a+}\frac{1}{2a+}\frac{1}{a+}\frac{1}{2a+}\cdots$

16. If $\frac{p}{q}$ be an even convergent and $\frac{X}{Y}$ be an odd convergent of a simple continued fraction whose value is F, prove that

(i) $F^2 > \frac{pX}{qY}$ if $\frac{X}{Y}$ preceds $\frac{p}{q}$,

(ii) $F^2 < \frac{pX}{qY}$ if $\frac{p}{q}$ preceds $\frac{X}{Y}$.

17. If $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$ are consecutive convergents of a simple continued fraction F prove that F^2 lies between $\frac{pp'}{qq'}$ and $\frac{p'p''}{q'q''}$.

18. If $\frac{p_n}{q_n}$ be the nth convergent of the recurring continued fraction $\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\cdots$ prove that

(i) $p_n - 4p_{n-2} + p_{n-4} = 0, n > 4$

(ii) $q_n - 4q_{n-2} + q_{n-4} = 0, n > 4$.

19. If $\frac{p_n}{q_n}$ be the nth convergent of the recurring continued fraction $a + \frac{1}{b+}\frac{1}{c+}$, prove that $p_{3n}q_3 - q_{3n}p_3 = q_{3n-3}$.

20. If $\frac{p_n}{q_n}$ be the nth convergent of the recurring continued fraction $\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}$, prove that $p_{3n+3} = bp_{3n} + (bc+1)q_{3n}$.

21. Express $\frac{1}{2}(\sqrt{5}+1)$ as a simple continued fraction.

If $\frac{p_r}{q_r}$ be the rth convergent of the continued fraction, prove that

(i) $p_{2n} = p_{2n-1} + p_{2n-2}$

(ii) $p_3 + p_5 + \cdots + p_{2n-1} = p_{2n} - p_2, n \geq 2$.

22. Express $\sqrt{17}$ as a simple continued fraction.

If $\frac{p_r}{q_r}$ be the rth convergent of the continued fraction, prove that

(i) $p_{n+1} + p_{n-1} = 34q_n$ for $n \geq 2$,

(ii) $q_{n+1} + q_{n-1} = 2p_n$ for $n \geq 2$,

(iii) $p_n^2 - 17q_n^2 = (-1)^n$ for $n \geq 1$.

23. If $x = \frac{1}{a+}\frac{1}{b+}$, $y = \frac{1}{2a+}\frac{1}{2b+}$, $z = \frac{1}{3a+}\frac{1}{3b+}$ prove that $x(y^2 - z^2) + 2y(z^2 - x^2) + 3z(x^2 - y^2) = 0$.

24. (i) If $x = a + \frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\cdots$, $y = \frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\cdots$ prove that $xy = \frac{a}{b}$.

(ii) If $x = \frac{1}{a+}\frac{1}{b+}\frac{1}{c+}$, $y = c + \frac{1}{b+}\frac{1}{a+}$ prove that $xy = \frac{1+bc}{1+ab}$.

(iii) If $x = \frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{d+}$, $y = d + \frac{1}{c+}\frac{1}{b+}\frac{1}{a+}$ prove that $xy = \frac{b+d+bcd}{a+c+abc}$.

25. If $\frac{p_n}{q_n}$ be the nth convergent of the simple continued fraction $\frac{1}{a+}\frac{1}{a+}\frac{1}{a+}\cdots$ show that $p_n = q_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ where α, β are the roots of the equation $x^2 - ax - 1 = 0$.

ANSWERS TO EXERCISES

Exercises 1B (page 32)

24. $x = \frac{2c}{49}, y = \frac{3c}{49}, z = \frac{6c}{49}$. **25.** (i) 4, (ii) 27. **32.** $x = \frac{a}{3}, y = \frac{b}{3}$.

33. (i) $3^8, x = 2$; (ii) $\frac{25}{24}, x = \frac{1}{12}$; (iii) $\frac{25}{8}, x = \frac{1}{4}, y = \frac{1}{2}$;

(iv) $\frac{1}{162}, x = \frac{1}{6}, y = \frac{1}{9}$; (v) $108, x = 2, y = 3$;

(vi) $\frac{1}{8}, x = y = \frac{1}{2}$; (vii) $2\sqrt{2}, x = y = z = \sqrt{2}$.

Exercises 2A (page 61)

11. $\frac{12}{5} + \frac{9}{5}i$. **13.** 5, 2.

19. (i) $\sqrt{2}, \frac{\pi}{2}$; (ii) $\frac{1}{\sqrt{2}}, -\frac{11\pi}{12}$; (iii) $-\sec\theta, -\pi + \theta$; (iv) $\operatorname{cosec}\theta, \frac{\pi}{2} - \theta$;

(v) $\cot\frac{\theta}{2}, -\frac{\pi}{2} + \theta$; (vi) $1, -\frac{\pi}{2} + \theta$; (vii) $-2\cos\theta, -\pi + \theta$;

(viii) $-2\cos\theta, \pi - \theta$; (ix) $\sin\theta, -\frac{\pi}{2} + \theta$; (x) $-\cos\theta, \pi - \theta$.

20. (i) $\tan\frac{\theta}{2}, -\frac{\pi}{2}$; (ii) $\operatorname{cosec}\frac{\theta}{2}, \frac{\pi}{2} - \frac{\theta}{2}$; (iii) $\sec\frac{\theta}{2}, -\frac{\theta}{2}$; (iv) $\operatorname{cosec}\theta, \frac{\pi}{2} - \theta$.

33. (i) $-2, 1 \pm \sqrt{3}i$; (ii) $-1, \pm i, \frac{1 \pm \sqrt{3}i}{2}$;

(iii) $\sqrt{2}(\cos\frac{2\pi}{6} \pm i\sin\frac{2\pi}{6}), \sqrt{2}(\cos\frac{8\pi}{6} \pm i\sin\frac{8\pi}{6})$;

(iv) $\cos\frac{(6k+2)\pi}{9} \pm i\sin\frac{(6k+2)\pi}{9}, k = 0, 1, 2$.

(v) $\frac{1}{2} + \frac{1}{2}i\cot\frac{(2k+1)\pi}{8}, k = 0, 1, \ldots, 8$. (vi) $-\frac{1}{2} + \frac{1}{2}i\cot\frac{k\pi}{8}, k = 1, 2, \ldots, 7$.

(vii) $\sqrt[6]{2}\{\cos(8k+3)\frac{\pi}{12} \pm i\sin(8k+3)\frac{\pi}{12}\}, k = 0, 1, 2$.

(viii) $i\cot\frac{k\pi}{6}, k = 1, 2, \ldots, 5$.

37. (i) $\cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}, k = 0, 1, 2, 3$.

(ii) $\cos(4k+1)\frac{\pi}{6} + i\sin(4k+1)\frac{\pi}{6}, k = 0, 1, 2$.

(iii) $\cos(4k-1)\frac{\pi}{8} + i\sin(4k-1)\frac{\pi}{8}, k = 0, 1, 2, 3$.

(iv) $\cos(4k+1)\frac{\pi}{8} + i\sin(4k+1)\frac{\pi}{8}, k = 0, 1, 2, 3$.

Exercises 2B (page 82)

2. (i) $\log 2 + (2n+1)\pi i$, (ii) $\log 4 + (4n+1)\frac{\pi}{2}i$,

(iii) $\frac{1}{2}\log 2 + (8n-1)\frac{\pi}{4}i$, (iv) $\frac{1}{3}\log 5 + (2n\pi + \tan^{-1}\frac{4}{3})i$.

5. (i) $2\log 2 + 2n\pi i, 2\log 2$; (ii) $2\log 2 + (2n+1)\pi i, 2\log 2 + \pi i$;
(iii) $2\log 2 + (4n+1)\frac{\pi}{2}i, 2\log 2 + \frac{\pi}{2}$; (iv) $2\log 2 + (4n-1)\frac{\pi}{2}i, 2\log 2 - \frac{\pi}{2}$.

7. (i) $0 + (\frac{\pi}{2} - \theta)i$, (ii) $0 + (\theta - \frac{\pi}{2})i$,
(iii) $\log \operatorname{cosec}\theta + (\frac{\pi}{2} - \theta)i$, (iv) $\log(-2\cos\theta) + i(\theta - \pi)$.

12. (i) 4; 4, (ii) $\sqrt{2}, -\sqrt{2}$; $\sqrt{2}$, (iii) $e^{\sqrt{2}\log 2}[\cos 2\sqrt{2}n\pi + i\sin 2\sqrt{2}n\pi]; e^{\sqrt{2}\log 2}$,
(iv) $e^{\sqrt{2}\log 2}[\cos\sqrt{2}(2n+1)\pi + i\sin\sqrt{2}(2n+1)\pi]; e^{\sqrt{2}\log 2}[\cos\sqrt{2}\pi + i\sin\sqrt{2}\pi]$.
(v) $e^{-2n\pi}\{\cos(\log 3) + i\sin(\log 3)\}, \cos(\log 3) + i\sin(\log 3)$;
(vi) $e^{-(2n+1)\pi}, e^{-\pi}$;
(vii) $e^{(1/2)\log 2 - (8n+1)(\pi/4)}\{\cos(2n\pi + \frac{\pi}{4} + \frac{1}{2}\log 2) + i\sin(2n\pi + \frac{\pi}{4} + \frac{1}{2}\log 2)\}$, the p.v. corresponds to $n = 0$;
(viii) $e^{-(4n+1)(\pi^2/8)}\{\cos[(4n+1)\frac{\pi}{4}\log 2] + i\sin[(4n+1)\frac{\pi}{4}\log 2]\}$, the p.v. corresponds to $n = 0$.

Exercises 2C (page 102)

7. (i) $n\pi + (-1)^n\frac{\pi}{6}$, (ii) $2n\pi \pm \frac{\pi}{3}$,
(iii) $n\pi + (-1)^n\log(\sqrt{5}+2)i$, (iv) $2n\pi \pm \{\frac{\pi}{2} - i\log(\sqrt{5}+2)\}$,
(v) $(2n\pi - \frac{\pi}{2}) \pm i\log(2+\sqrt{3})$, (vi) $(2n+1)\pi \pm i\log(2-\sqrt{3})$,
(vii) $2n\pi i + (-1)^n\log(\sqrt{5}+2)$, (viii) $2n\pi i \pm \log(2+\sqrt{3})$.

Exercises 3A (page 138)

9. (i) 2, −1; (ii) 37, −6;

10. (i) −22, 7; (ii) 10, −4; (iii) 12, 6.

12. (i) $x = 27t - 10, y = 8t - 3, t = 0, \pm1, \pm2, \cdots; x = 17, y = 5$.
(ii) $x = 17t + 7, y = 12t + 5, t = 0, \pm1, \pm2, \cdots; x = 7, y = 5$.
(iii) $x = 13t + 30, y = 35t + 80, t = 0, \pm1, \pm2, \cdots; x = 4, y = 10$.
(iv) $x = 17t + 40, y = 41t + 96, t = 0, \pm1, \pm2, \cdots; x = 6, y = 14$.
(v) $x = 13t + 45, y = 29t + 100, t = 0, \pm1, \pm2, \cdots; x = 6, y = 13$.
(vi) $x = 55t - 7, y = 63t - 8, t = 0, \pm1, \pm2, \cdots; x = 48, y = 55$.

13. (i) 1, 20; 8, 9. (ii) 3, 19; 7, 6. (iii) 4, 11; 29, 2. (iv) 2, 55; 7, 13; 12,1.

14. (i) 181, 19; 136, 64; 91, 109; 46,154. (ii) 33, 67; 78, 22.

18. (i) 15, (ii) 18, (iii) 16, (iv) 24.

19. (i) 156, (ii) 576, (iii) 1767, (iv) 9576.

20. (i) 3224, 19344; (ii) 403, 12090. **21.** 360.

Exercises 3B (page 157)

1. (i) 3, 20, 37, 54, 71, 88; (ii) 10, 27, 44, 61, 78, 95,

3. (i) 2 or 9, (ii) 0 or 9, (iii) 3, (iv) 7. **6.** (i) 0, (ii) 0.

9. (i) $x \equiv 2 (\mathrm{mod}\ 5)$, (ii) $x \equiv 9 (\mathrm{mod}\ 15)$, (iii) $x \equiv 4, 11, 18, 25, 32 (\mathrm{mod}\ 35)$, (iv) $x \equiv 6, 21, 36, 51, 66, 81, 96 (\mathrm{mod}\ 105)$.

10. (i) $x \equiv 18 (\mathrm{mod}\ 35)$, (ii) $x \equiv 53 (\mathrm{mod}\ 105)$, (iii) $x \equiv 157 (\mathrm{mod}\ 280)$

11. (i) 47, (ii)158.

12. (i) $160 + 315t, t = 0, \pm1, \pm2, \ldots$ (ii) $1104 + 1260t, t = 0, \pm1, \pm2, \ldots$

13. (i) $x \equiv 43$ (mod 63), (ii) $x \equiv 101$ (mod 108).

Exercises 3C (page 170)

1. (i) 2^{10}, (ii) 1152, (iii) 1920.

17. (i) 1, (ii) 9. **19.** (i) 3, (ii) 7. **20.** (i) 01, (ii) 61.

Exercises 3D (page 184)

1. (i) 74, (ii) 37 , (iii) 98. **2.** (i) 53, (ii) 81.

Exercises 4 (page 193)

1. (i) 1, (ii) 1. **2.** (i) $x^5 - x^4 + x^3, 1$; (ii) $x^3 + 3x^2 - x + 1, 3$.

4. (i) $(x-1)^5 - 10(x-1)^3 - 8(x-1)^2 + 9(x-1) + 7$,
(ii) $x^4 + 9x^3 + 37x^2 + 87x + 90$, (iii) $h = -2$.

5. (i) $-2x + 1$, (ii) $10x - 10$, (iii) $6x + 1$, (iv) $x^3 + x$, (v) $2x^2 + 3x + 3$.

6. $x^2 + 4x - 2$, **7.** 30.

Exercises 5A (page 208)

1. (i) 4, (ii) 19, (iii) 4, (iv) 16.

3. (i) $-1, -1, \pm i$; (ii) $-1, \frac{-1\pm\sqrt{3}i}{2}, \frac{-1\pm\sqrt{3}i}{2}$; (iii) $\pm\sqrt{2}i, \frac{-1\pm\sqrt{3}i}{2}, \frac{-1\pm\sqrt{3}i}{2}$.

6. $k = -20; 2, 2, 5$ and $k = -16; 4, 4, 1$.

7. $a = 1; 1, 1, 4$ and $a = \frac{5}{4}, 2, 2, \frac{4}{5}$. **8.** $3; 1, 1, 1, \frac{-3\pm\sqrt{5}}{2}$.

9. (i) $1 \pm i, \frac{-1\pm\sqrt{7}i}{2}$; (ii) $2 \pm i, 1, \frac{-1\pm\sqrt{3}i}{2}$; (iii) $1 \pm \sqrt{2}, \frac{-1\pm\sqrt{7}i}{4}$;
(iv) $3, -2, \sqrt{2} \pm i, -2 \pm i$; (v) $\frac{-3\pm\sqrt{7}i}{2}, 1 \pm i$; (vi) $-\frac{1}{3} \pm \frac{2\sqrt{2}}{3}i, \pm\sqrt{2}i$.

10. $x^4 - 14x^2 + 1 = 0$. **11.** $x^4 + 6x^2 + 25 = 0$.

12. $-1, \frac{-1\pm2\sqrt{2}i}{3}$. **13.** $3, -1 \pm 2\sqrt{2}i$. **14.** $\frac{1\pm2\sqrt{2}i}{3}, \frac{-1\pm\sqrt{3}i}{2}$.

15. $\pm3i, 1 \pm 2\sqrt{2}i$. **19.** $(-4, -3), (-2, -1), (3, 5)$.

20. $(-\infty, -\frac{3}{2}), (-1, -\frac{1}{2}), (\frac{1}{2}, 1), (\frac{3}{2}, \frac{7}{4})$.

21. When $k = 117$, two equal roots $-3, -3$; when $k = -8$, two equal roots 2, 2 and two distinct real roots in $(-\infty, -3), (-3, 1)$; when $k = -11$, two equal roots 1, 1 and two distinct real roots in $(\infty, -3), (2, \infty)$.

22. No real root if $k > 64$, four real roots if $-17 \leq k \leq 64$, two real roots if $k < -17$. **23.** $-13 < r < -8$.

24. (i) $-2, 3, \frac{1 \pm \sqrt{11}i}{2}$; (ii) $2, -5, \frac{-3 \pm \sqrt{5}}{2}$; (iii) $2, 3, \frac{3 \pm \sqrt{7}i}{2}$.

Exercises 5B (page 221)

1. (i) 3, (ii) 1, (iii) 3.

2. (i) $x^3 - 3x - 1, x^2 - 1, 2x + 1, 1; (-2, -1), (-1, 0), (1, 2)$.

(ii) $x^3 - 7x + 7, 3x^2 - 7, 2x - 3, 1; (-4, -3), (1, \frac{3}{2}), (\frac{3}{2}, 2)$.

(iii) $x^4 - x^2 - 2x - 5, 2x^3 - x - 1, x^2 + 3x + 10; (-2, -1), (1, 2)$.

(iv) $x^5 - 5x + 5, x^4 - 1, -1; (-2, -1)$.

5. (i) One positive, one negative and two complex roots;

(ii) All complex roots;

(iii) One positive, one negative and eight complex roots;

(iv) Five real roots and two complex roots.

6. (i) Two, (ii) Four, (iii) Two. **7.** One positive and two negative roots.

8. (b) (i) & (ii) One positive and two negative roots;

(iii), (iv) & (v) two positive and two negative roots.

Exercises 5C (page 230)

1. (i) $-6, \pm\sqrt{3}$; (ii) $1 \pm \sqrt{6}i, \pm\sqrt{3}$; (iii) $-2 \pm \sqrt{3}, \pm\frac{1}{\sqrt{2}}i$.

2. (i) $-2, \frac{-3 \pm \sqrt{5}}{2}$; (ii) $\frac{-1 \pm \sqrt{3}i}{2}, \frac{-1 \pm \sqrt{11}i}{2}$; (iii) $2 \pm \sqrt{3}, \frac{-5 \pm \sqrt{15}}{2}$.

3. (i) $-1, -2, -3$; (ii) $-2, -\frac{1}{2}, 1, \frac{5}{2}$; (iii) $-2, -\frac{3}{2}, -1, -\frac{1}{2}$.

4. (i) $\frac{2}{3}, 2, 6$; (ii) $-1, 2, -4, 8$; (iii) $-1 \pm \sqrt{2}i, \frac{1 \pm \sqrt{11}i}{6}$; (iv) $\frac{1}{3}, -1, 3, -9$.

5. (i) $-1, \pm\sqrt{2}i, \frac{1 \pm \sqrt{11}i}{6}$; (ii) $-3 \pm \sqrt{6}, -2 \pm \sqrt{5}$.

6. (i) $1, -2, 4, -5$; (ii) $1, 3, 2 \pm \sqrt{2}$.

7. (i) $1, -3, \frac{-1 \pm \sqrt{13}}{2}$, (ii) $1 \pm \sqrt{2}i, \frac{-5 \pm \sqrt{23}i}{4}$.

8. (i) $1, 2, 3, 6$; (ii) $1, 3, -3 \pm \sqrt{6}$, (iii) $1, 2, -\frac{1}{2}, -4$.

9. (i) $k = 2, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$; (ii) $k = 14, \pm 1, 3, 5$.

10. (i) $2b^3 - 3abc + a^2d = 0$, (ii) $b^3d = c^3a$, (iii) $2c^3 - 3bcd + ad^2 = 0$.

11. (i) $r^2 + p^2s - pqr = 0$, (ii) $p^3 + 8r = 4pq$, (iii) $r^2 + s(p^2 - 4q) = 0$,

(iv) $(p + r)(ps + r) + (q + s + 1)(s - 1)^2 = 0$, (v) $r^2 = p^2s$.

12. (i) $2 \pm 2i, -3$; (ii) $2 \pm 2i, 3$.

13. (i) $1 \pm i, -3 \pm 3i$; (ii) $-1 \pm i, -2 \pm 2i$.

14. (i) $\frac{-1 \pm \sqrt{7}i}{2}, \frac{-1 \pm \sqrt{7}i}{2}$; (ii) $-3, -3, \frac{1}{2}, \frac{1}{2}$.

15. (b) $1, 2, 3, 6$; (c) (i) $-1, 2, -4, 8$; (ii) $\frac{1}{3}, -1, 3, -9$.

16. (i) $q^2 - 2pr$, (ii) $q^3 - 3pqr + 3r^2$, (iii) $-p^3 = 24r$,

(iv) $p^4 - 4p^2q + 2q^2 + 4pr$, (v) $-qr$, (vi) $9r - pq$,

(vii) $2q^2 - 6pr$, (viii) $2p^3 - 9pq + 27r$.

17. (i) $ps + 2qr - p^2r$, (ii) $r^2 - 2qs$, (iii) $pr - 16s$,

(iv) $3r^2 - 8qs$, (v) $-p^3 + 4pq - 8r$.

19. (i) $\frac{pr - 4s}{s}$, (ii) $\frac{3ps - qr}{s}$, (iii) $\frac{qr^2 - 2q^2s - prs + 4s^2}{s^2}$, (iv) $\frac{ps - p^2r + 2qr}{s}$.

Exercises 5D (page 239)

1. (i) $m = 10$, (ii) $m = 6$.

2. (i) $\frac{a_2^2 - 2a_1a_3}{a_3^2}$, (ii) $\frac{a_1^2 - 2a_0a_2}{a_3^2}$, (iii) $\frac{3a_1a_2a_3 - a_2^3 - 3a_0a_3^2}{a_3^3}$.

5. (i) $-\frac{18}{5}$, (ii) $\frac{31}{5}$, (iii) $-\frac{42}{5}$.

6. (i) $(ry + q)^3 - 2q(ry + q)^2 + 4pr(ry + q) - 8r^2 = 0$,

(ii) $ry^3 + q(1 - r)y^2 + p(1 - r)^2y + (1 - r)^3 = 0$,

(iii) $r(y + p)^3 - q^2(y + p)^2 + pq^2(y + p) - q^3 = 0$,

(iv) $r(y + 1)^3 - pq(y + 1)^2 + p^3(y + 1) - p^3 = 0$.

7. (i) $y^3 - 2qy^2 + q^2y + r^2 = 0$, (ii) $(y + 2q)^3 - 3q(y + 2q)^2 - 27r^2 = 0$,

(iii) $y^3 + 4qy^2 + 5q^2y + 2q^3 + r^2 = 0$, (iv) $y^3 + 9qy - 27r = 0$.

8. $r^2(y + 1)^3 + q^3(y + 2) = 0$.

9. $(2y - 3)^3 - 2(2y - 3)^2(3y - 2) + 3(2y - 3)(3y - 2)^2 - (3y - 2)^3 = 0$.

10. (i) $\frac{3}{2}, \frac{3}{4}, \frac{1}{2}$; (ii) $2, \frac{2}{3}, \frac{2}{5}, -2$; (iii) $-\frac{1}{6}, -\frac{1}{2}, 1, \frac{1}{4}$.

11. $\pm\frac{1}{2}, \frac{-1 \pm \sqrt{3}i}{2}$. **12.** (i) $-1, -3, -\frac{2}{3}$; (ii) $2, 3, \frac{-1 \pm \sqrt{3}i}{2}$.

13. (i) $2, 3, -\frac{1}{2}$; (ii) $2, -\frac{1}{2}, \pm i$. **14.** $-5, \frac{-1 \pm 3\sqrt{3}i}{2}$.

15. $-1 \pm \sqrt{2}, -1 \pm \sqrt{5}i$. **16.** $a_0a_2 = a_1^2$.

17. $2a_1^3 - 3a_0a_1a_2 + a_0^2a_3 = 0$. **18.** Two positive and one negative roots.

19. Two real and two imaginary roots.

20. $y^4 + 3y^3 + 4y^2 + 3y + 1 = 0$. **22.** $y^3 + 33y^2 + 12y + 8 = 0$.

24. $y^3 + (p^2 - 3q)y^2 + (3q^2 - p^2q)y + (p^3r - q^3) = 0, p^3r = q^3$.

25. $y^3 + (9q - 3p^2)y - (2p^3 - 9pq + 27r) = 0, 2p^3 - 9pq + 27r = 0$.

Exercises 5E (page 260)

1. (i) $\pm i, \frac{1}{2}(-1 \pm \sqrt{3}i)$; (ii) $\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(5 \pm \sqrt{21})$;

(iii) $\frac{2+\sqrt{3}\pm\sqrt{3+2\sqrt{3}}}{2}, \frac{2-\sqrt{3}\pm\sqrt{3-2\sqrt{3}}}{2}$; (iv) $1, \frac{1}{2}(-3 \pm \sqrt{5}), \frac{1}{4}(-1 \pm \sqrt{15}i)$;

(v) $1, \pm i, -3, -\frac{1}{3}$; (vi) $-1, 2, \frac{1}{2}, \pm i$;

(vii) $\pm 1, \frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(-3 \pm \sqrt{5})$; (viii) $\pm i, \pm(3 + \sqrt{5})i, \pm(3 - \sqrt{5})i$;

(ix) $1, 1, \pm i, \frac{1}{2}(-1 \pm \sqrt{3}i)$; (x) $\pm 1, \pm i, -1, \frac{1}{2}(-3 \pm \sqrt{5})$.

2. (i) $\frac{1}{2}(-1 \pm \sqrt{3}i), \frac{1}{2}(-1 \pm \sqrt{3}i)$; (ii) $-1, \pm i, \frac{1}{4}(3 \pm \sqrt{7}i)$;

(iii) $\pm i, \pm i, \frac{1}{4}(3 \pm \sqrt{7}i)$, (iv) $\frac{1}{2}(\pm\sqrt{6} \pm \sqrt{2})i, \frac{1}{2\sqrt{3}}(\pm\sqrt{14} \pm \sqrt{2})i$.

3. $m = 2, n = 1; m = -2, n = 1; 1 \pm 2i, -2, -\frac{1}{3}$.

4. $m = \pm 2; \frac{-3\pm\sqrt{7}i}{2}, -1 \pm \sqrt{3}i$. **5.** $1 \pm i, 3 \pm \sqrt{3}$.

6. $-2 \pm i, \frac{-3\pm\sqrt{3}i}{2}$. **11.** (i) $x^3 + 2x^2 - x + 2 = 0$.

(ii) $x^3 + 2x^2 + 5x + 8 = 0$. (iii) $x^3 + 4x^2 - 4x + 1 = 0$.

12. $x^3 + 3x^2 + 30x + 55 = 0$. **13.** (i) $x^4 + x^3 + 2x^2 + x + 2 = 0$,

(ii) $x^4 + 2x^3 + 7x^2 - 5x + 3 = 0$, (iii) $x^4 + x^3 + 7x^2 + 14x + 9 = 0$.

22. (i) 12, (ii) 16, (iii) 24, (iv) 24.

Exercises 5F (page 269)

1. (i) $y^3 + 18y^2 + 81y + 135 = 0$, (ii) $y^3 - 18y^2 + 81y = 0$,

(iii) $y^3 - 162y^2 + 6561y = 0$.

2. (i) One real root, (ii) two equal roots, (iii) two equal roots.

8. (i) $27r^2 = 2q^3$, (ii) $216r^2 = 5q^3$.

10. (i) -3, -3, 6; (ii) $-4, 2 \pm \sqrt{3}i$; (iii) $4\cos\frac{2\pi}{9}, 4\cos\frac{8\pi}{9}, 4\cos\frac{14\pi}{9}$;

(iv) $2\cos\frac{A}{3}, 2\cos\frac{2\pi+A}{3}, 2\cos\frac{4\pi+A}{3}$; (v) $2, 2\sqrt{2}\cos\frac{5\pi}{12}, 2\sqrt{2}\cos\frac{11\pi}{12}$;

(vi) $\frac{1}{3}(\sqrt[3]{3} + \sqrt[3]{9}), \frac{1}{3}(\omega\sqrt[3]{3} + \omega^2\sqrt[3]{9}), \frac{1}{3}(\omega^2\sqrt[3]{3} + \omega\sqrt[3]{9})$;

(vii) $1, \sqrt{2}\cos\frac{5\pi}{12}, \sqrt{2}\cos\frac{11\pi}{12}$; (viii) -5, -5, 1;

(ix) $7, \frac{1}{2}(-1 \pm \sqrt{3}i)$; (x) $2\cos\frac{\pi}{9} - 1, 2\cos\frac{5\pi}{9} - 1, 2\cos\frac{7\pi}{9} - 1$.

11. (i) $-\frac{1}{5}, \frac{1\pm\sqrt{5}}{10}$; (ii) $\frac{3}{4}, \frac{-3\pm3\sqrt{5}}{8}$; (iii) $\frac{1}{3}, \frac{1\pm\sqrt{5}}{6}$; (iv) $-\frac{3}{2}, \frac{3\pm3\sqrt{5}}{4}$.

12. (i) 2, 2, -4; (ii) $-5, -4\omega - \omega^2, -4\omega^2 - \omega$.

Exercises 5G (page 279)

1. (i) $1 \pm 2i, -1 \pm \sqrt{2}$; (ii) $1 \pm 3i, -1 \pm \sqrt{7}$; (iii) $\frac{-3\pm\sqrt{7}i}{2}, \frac{1\pm\sqrt{11}i}{2}$;

(iv) $1 \pm \sqrt{2}i, -1 \pm \sqrt{6}$; (v) $-1 \pm \sqrt{2}, 1 \pm \sqrt{2}i$; (vi) $\frac{-1\pm\sqrt{7}i}{2}, \frac{1\pm\sqrt{19}i}{2}$;

(vii) $1 \pm 3i, -1 \pm 2i, \lambda = \frac{15}{2}$; (viii) 1, 2, -2, -3;

(ix) $-3 \pm \sqrt{7}, 1 \pm \sqrt{3}i$, (x) $-2 \pm \sqrt{2}, -4 \pm 2i$,

(xi) $-1 \pm i, \frac{-1\pm\sqrt{3}i}{2}$, (xii) $-2 \pm \sqrt{2}, \frac{1\pm i}{2}$.

2. (i) $1 \pm 2i, -1 \pm \sqrt{2}$, (ii) $-1 \pm \sqrt{2}, 1 \pm \sqrt{2}i$, (iii) $-1 \pm \sqrt{2}, -1, 3$,

(iv) $2 \pm \sqrt{2}, -2 \pm \sqrt{2}$, (v) $1 \pm \sqrt{6}, -1 \pm \sqrt{2}i$.

3. (i) $1 \pm \sqrt{2}, -1 \pm i$, (ii) $1 \pm \sqrt{2}i, -1 \pm \sqrt{6}$, (iii) $-1 \pm \sqrt{2}, 1 \pm \sqrt{2}i$.

Exercises 6A (page 289)

1. (i) $\frac{1}{4}n(n+1)(n+2)(n+3)$,

(ii) $\frac{1}{5}(n+1)(n+2)(n+3)(n+4)(n+5) - 24$,

(iii) $\frac{1}{12}(3n-1)(3n+2)(3n+5)(3n+8) + \frac{20}{3}$,

(iv) $\frac{1}{12}n(n+1)(n+2)(3n+5)$, (v) $\frac{1}{3}n(n+1)(n+2) + n(n+1)$,

(vi) $\frac{1}{5}n(n+1)(n+2)(n+3)(n+4) - \frac{1}{4}n(n+1)(n+2)(n+3)$,

(vii) $\frac{1}{4}n(n+1)(n+4)(n+5)$, (viii) $\frac{1}{3}n(n+1)(4n-1)$.

2. (i) $\frac{1}{2} - \frac{1}{n+2}$, (ii) $\frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$, (iii) $\frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$,

(iv) $\frac{17n^3+66n^2+61n}{36(n+1)(n+2)(n+3)}$, (v) $\frac{5n^2+n}{12(n+2)(n+3)}$, (vi) $\frac{2}{3} - \frac{2.4.6...(2n+2)}{3.5.7...(2n+3)}$

(vii) $\frac{1.3.5...(2n+3)}{2.4.6...(2n+2)} - \frac{3}{2}$, (viii) $\frac{2.5.8...(3n+2)}{1.4.7...(3n+1)} - 2$.

3. (i) $\frac{\sin\frac{n(\pi+\theta)}{2}}{\sin\frac{\pi+\theta}{2}} . \sin\frac{(n+1)(\pi+\theta)}{2}$, (ii) $\frac{n}{2} - \frac{1}{2}\frac{\sin n\theta}{\sin\theta}\cos(n+1)\theta$,

(iii) $\frac{n}{2} + \frac{1}{2}\frac{\sin n\theta}{\sin\theta}\cos(n+1)\theta$, (iv) $\frac{n}{2}\cos\theta - \frac{1}{2}\{\frac{\sin n\theta}{\sin\theta}\cos(n+2)\theta\}$,

(v) $\frac{n}{2}\cos\theta + \frac{1}{2}\frac{\sin 2n\theta}{\sin 2\theta}\cos(2n+1)\theta$, (vi) $\frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - 2\cot 2\theta$,

(vii) $\cot x\tan(n+1)x - n - 1$, (viii) $\mathrm{cosec}\theta\{\tan\theta - \tan(n+1)\theta\}$,

(ix) $\mathrm{cosec}^2\theta - 2^n\mathrm{cosec}^2(2^n\theta)$, (x) $\frac{1}{4}\mathrm{cosec}\frac{\theta}{2}[\sec(2n+1)\frac{\theta}{2} - \sec\frac{\theta}{2}]$,

(xi) $\log\sin 2^n\theta - \log\sin\theta$, (xii) $\frac{1}{2}(\tan 3^n x - \tan x)$,

(xiii) $\tan^{-1}(2n+1) - \tan^{-1}1$, (xiv) $\tan^{-1}(n+1) - \tan^{-1}1$,

(xv) $\tan^{-1}\frac{n+1}{2} - \tan^{-1}\frac{1}{2}$, (xiv) $\log\sin\frac{3^n\theta}{2} - \log\sin\frac{1}{2}\theta$.

Exercises 6B (page 304)

1. (i) $n^2+n+3, \frac{n}{3}(n^2+3n+11)$; (ii) $n^3+n, \frac{n}{4}(n+1)(n^2+n+2)$;

(iii) $n(n+1)(n+5), \frac{n}{4}(n+1)(n+2)(n+7)$;

(iv) $3^n+2n+4, \frac{3}{2}(3^n-1)+n^2+5n$;

(v) $2^{n+1}-\frac{n^2+n}{2}, 4(2^n-1)-\frac{1}{6}(n+1)(n+2)$;

(vi) $3^{n-1}+n+1, \frac{3^n-1}{2}+\frac{n(n+3)}{2}$; (vii) $\frac{n^2+1}{n(n+1)}\cdot 2^{n-1}, \frac{n.2^n}{n+1}$;

(viii) $\frac{(n+2)}{n(n+1)}\cdot\frac{1}{2^{n+2}}, \frac{1}{2}-\frac{1}{(n+1)2^{n+1}}$; (ix) $(2n+1).3^{n-1}, n.3^n$.

2. (i) $u_r = 3^{r-1}+2^{r-1}, \frac{1}{2}(2.2^n+3^n-3)$; (ii) $u_r = 2^{r-1}+4^{r-1}$,

$\frac{1}{3}(3.2^n+4^n-4)$; (iii) $u_r = (r+1)2^r, n.2^{n+1}$;

(iv) $u_r = 2^r+(-1)^r+2, 2^{n+1}+\frac{1}{2}(-1)^n+2n-\frac{5}{2}$.

Exercises 6C (page 306)

1. (i) $e^{\cos\theta}(\sin\theta)$, (ii) $\frac{1-c\cos\theta}{1-2c\cos\theta+c^2}$, (iii) $\frac{\cos\theta-x}{1-2x\cos\theta+x^2}$,

(iv) $e^{x\cos\theta}\cos(x\sin\theta)$, (v) $\log(2\cos\theta)$, (vi) $\frac{\pi}{2}-\theta$,

(vii) $\frac{\pi}{2}-\theta$, (viii) $\frac{1}{2}\log(4\cos\theta)$, (ix) 0, (x) $\frac{1}{2}\log(\sec\theta)$.

Exercises 7 (page 328)

1. (i) $2+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3}$, (ii) $2+\frac{1}{1+}\frac{1}{5+}\frac{1}{3+}\frac{1}{1}$, (iii) $2+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{7}$, (iv) $0+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{4}$.

2. (i) $2+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1}$, (ii) $2+\frac{1}{1+}\frac{1}{5+}\frac{1}{4}$, (iii) $2+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\frac{1}{1}$, (iv) $0+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1}$.

3. (i) $m=11, n=8$, (ii) $m=4, n=3$ (iii) $m=25, n=16$.

4. (i) $x=15t+11, y=41t+30$; (ii) $x=15t+4, y=41t+11$;

(iii) $x=15t+11, y=-41t-30$; (iv) $x=15t+4, y=-41t-11$;

(v) $x=19t+15, y=14t+11$; (vi) $x=19t+4, y=14t+3$;

(vii) $x=19t+15, y=-14t-11$; (viii) $x=19t+4, y=-14t-3$; where $t=0,\pm1,\pm2,\ldots$

5. (i) $1+\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\cdots$, (ii) $2+\frac{1}{\overset{\bullet}{1}+}\frac{1}{1+}\frac{1}{1+}\frac{1}{\overset{\bullet}{4}+}$, (iii) $3+\frac{1}{\overset{\bullet}{3}+}\frac{1}{\overset{\bullet}{6}+}\cdots$,

(iv) $3+\frac{1}{\overset{\bullet}{1}+}\frac{1}{2+}\frac{1}{1+}\frac{1}{\overset{\bullet}{6}+}$, (v) $1+\frac{1}{1+}\frac{1}{\overset{\bullet}{1}+}\frac{1}{3+}\frac{1}{\overset{\bullet}{2}+}$, (vi) $\frac{1}{\overset{\bullet}{1}}+\frac{1}{2+}\frac{1}{\overset{\bullet}{3}+}$

6. $\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$

BIBLIOGRAPHY

Barnard, S.& Child, J.M. — *Higher Algebra* Macmillan & Co., London,1967

Burnside, W.S.& Panton, A.W. — *Theory of Equations*, volume 1 S. Chand & Co., 1960

Burton, D. M. — *Elementary Number Theory* Universal Book Stall, New Delhi,2003

Churchill, R.V. — *Complex Variables and Applications* McGraw-Hill Book Company, Inc.1960

Durell, C.V.& Robson, A.R. — *Advanced Trigonometry* G. Bell & Sons Ltd, London,1950

Hall, H.S.& Knight, S.R. — *Higher Algebra* Macmillan & Company,1960

Hardy, G.H. — *Pure Mathematics* Universal Book Stall, New Delhi,1989

Niven, I. Zuckerman, H.S. Montgomery, H.L. — *An Introduction to the Theory of Numbers* John Wiley & Sons, Inc.,2004

Stewert, B.M. — *Theory of Numbers* The Macmillan Company, NY,1958